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MATHEMATICS

THE MINNEAPOLIS JOURNAL EDITION

NEW YORK

THE DOUBLEDAY & McCLURE CO.

1900

Norwood Press
J. S. Cushing & Co. — Berwick & Smith
Norwood Mass. U.S.A.

PREFACE

This book is essentially practical. It is intended for young men and others who want to obtain such a knowledge of mathematics as shall be of service to them in their business as mechanics or engineers, and to obtain it by study at home, without the advantages — and without the expense — of outside assistance.

The book covers a pretty wide field, and no young man can master it without finding himself greatly benefited — that is to say, greatly strengthened and armored for the battle of life.

And yet there is nothing in the book which may not be mastered by a little diligent study pursued at home, day by day, say for a year.

In some of the topics a few principles, taken from the higher mathematics, have been assumed without proofs. The proofs of them would be too difficult for useful insertion here. But these assumed principles are so presented that their reasonableness can be very well inferred, although the proofs for them be not mathematically established.

Each part in the book is complete in itself, and therefore may be taken up without a knowledge of the other parts. But it will be better if *all* the parts be mastered before the book is laid aside. Especially is the part entitled

"Elementary Algebra" recommended. A knowledge of the solution of simple equations, for example, is one of those things which no one needing to make calculations can possibly afford to do without.

The "First Course in Geometry" is also especially recommended. It is a course in pure reasoning based upon a method of study that has been pursued with immense advantage to mankind for over two thousand years. Also, the results established in the course will be of great help in throwing light on processes made use of in other parts of the book. But especially is the course commended to students for the insight that it will give them as to the rigorous way in which all reasoning in mathematics is pursued when the study is taken up under suitable conditions.

CONTENTS

LIST OF ABBREVIATIONS

bbl. = barrel	lb. = pound
bd. ft. = board foot	lk. = link
B. M. = board measure	M. = thousand
bu. = bushel	mi. = mile
ch. = chain	oz. = ounce
circum. = circumference	P. = power
cu. ft. = cubic foot	rd. = rod
cu. in. = cubic inch	sq. ch. = square chain
cu. yd. = cubic yard	sq. ft. = square foot
diam. = diameter	sq. in. = square inch
doz. = dozen	sq. mi. = square mile
ft. = foot	sq. rd. = square rod
gal. = gallon	sq. yd. = square yard
H. C. F. = highest common factor	W. = weight
in. = inch	yd. = yard

I

MECHANICS' BIDS AND ESTIMATES

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INTRODUCTORY NOTE

That this course may be of the largest value to mechanics, contractors, and others whose school training has been limited, it is necessary to begin with a thorough course in industrial arithmetic. Upon no other educational foundation can you expect to build intelligent and satisfactory bids and estimates. The work begins very easy. You should do every exercise. The answers, with hints for solution whenever necessary, are given at the end of the course.

Lesson No. 1. Simple Linear Measurements

TABLE

320 rods = 1 mile
1760 yards = 1 mile
$16\frac{1}{2}$ feet = 1 rod

EXERCISES

1. Find the cost of 20 mi. of wire at 35¢ a pound, supposing that 1 lb. stretches 50 ft.

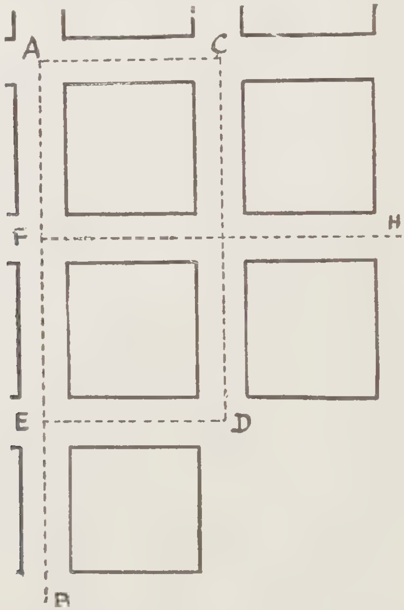
2. What is the cost of a cable 921 ft. long at 95¢ a yard?

3. A township is 6 mi. square. Find the cost of fencing it at 65¢ a rod.

4. How many inches are there in $13\frac{1}{2}$ rd.?

5. A city is laid out in squares, each of which measures one-eighth of a mile from the center of the street crossing

to the center of the next street crossing. Find the cost of laying water pipes along the streets, as shown by the dotted lines in the diagram, at an expense of \$ 1.15 per foot.



Lesson No. 2. Rods, Chains, and Miles

TABLE

1 chain = 4 rods
1 chain = 66 feet
80 chains = 1 mile

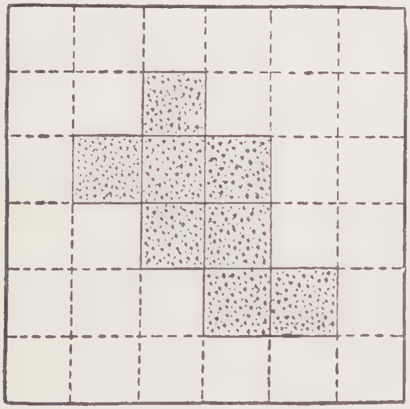
EXERCISES

1. How many yards are there in 24 ch.?

2. Find the distance in yards around a field 24 ch. long by 28 rd. wide.

3. How many chains in 20,460 ft.?

4. A surveyor reports the length of a projected railway as 6560 ch. What will it cost to build the road at an estimate of $ 20,000 a mile?



5. The above drawing represents a Dakota township, and the shaded portion a wheat farm belonging to one man. The township is 6 mi. square. Find the total cost of fencing in the farm, as per the lines in the drawing, at $ 5.70 per chain.

Lesson No. 3. Simple Square Measurements

TABLE

144 square inches = a square foot.
9 square feet = a square yard.

RULE: *To find the area of any rectangle, — that is, any four-sided figure the corners of which are right angles, — multiply the length by the breadth. Before multiplying you must have the length and breadth both in the same denomination. If both are inches, the answer will be square inches; if both are feet, the answer will be square feet; and so on.*

EXERCISES

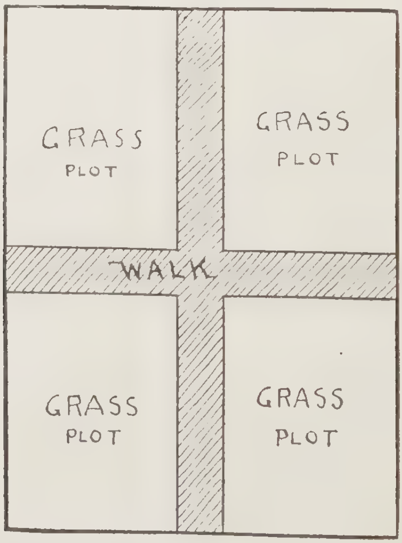
1. Find the cost of covering a square 28 in. by 14 in. with gold leaf at 7¢ a square inch.

2. How many square inches are there in $5\frac{1}{2}$ sq. yd.?

3. Find the cost of 18 sq. yd. of canvas at $12\frac{1}{2}$¢ a square foot.

4. A garden plot is 9 rd. long by 20 yd. wide. Find the cost of sodding it at $18\frac{1}{2}$¢ a square yard.

NOTE. — The rods must be reduced to yards before multiplying; the result after multiplying will be square yards.



5. An athletic field is 36 rd. by 27 rd.; it is surrounded by a tight board fence 12 ft. high. Find the cost of painting the fence, both sides, at $4\frac{1}{2}$¢ a square yard.

6. Find the cost of an inlaid floor $18\frac{1}{2}$ ft. by 16 ft. at $2.50 per square foot.

7. How many square yards are there in 16 sq. rd.?

NOTE. — There are $5\frac{1}{2}$ yd. in a rod. A square rod will equal $5\frac{1}{2} \times 5\frac{1}{2}$ sq. yd.

8. A road $2\frac{1}{2}$ mi. long is to be paved with asphalt. The asphalt is to reach from curb to curb, a distance of one chain. Find the expense of paving at $ 2.30 per square yard.

9. How many square yards in a square chain?

10. A rectangular garden plot, 51 yd. long by 39 yd. wide, is to be laid out with a gravel walk, as shown in the foregoing diagram, 6 ft. wide. The leveling of the entire plot will cost 3¢ a square yard; the walk will cost 8¢ a square yard; the sodding of the four sections will cost 12¢ a square yard; and an iron fence surrounding the garden will cost 75¢ a foot. Find the entire expense.

Lesson No. 4. Square Rods, Square Chains, and Square Miles

TABLE

160 square rods = an acre
10 square chains = an acre
640 acres = a square mile

EXERCISES

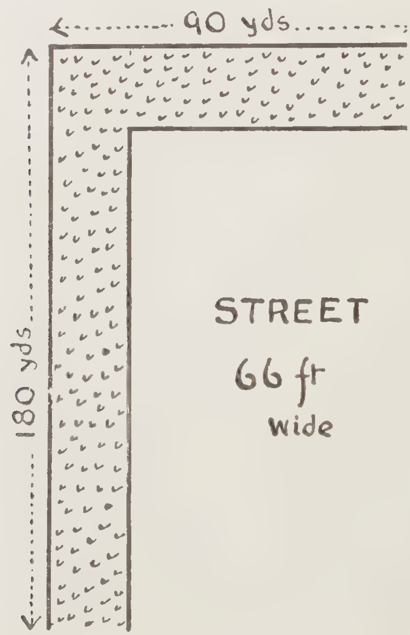
1. A field is 80 rd. long by 64 rd. wide. How many acres?

2. A piece of timber land is 20 mi. long by $3\frac{1}{2}$ mi. wide. What is it worth at $ 14.50 an acre?

3. A wheat field is $\frac{1}{2}$ mi. long and 12 ch. wide; the yield is 24 bu. to the acre. What is the wheat worth at 85¢ a bushel?

4. A square mile of land is divided into 40-acre fields, and each field is surrounded by a wire fence. Find the expense of fencing at $ 1.20 per rod.

5. Some city property is bought for the purpose of making a new street. The street is to be 66 ft. wide, and the two long sides as shown in the diagram measure 180 yd. and 90 yd. respectively. Find the cost of the property at 47¢ a square foot.



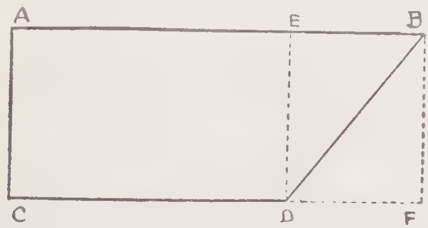
Lesson No. 5. Areas of Four-sided Figures, having two Adjacent Right Angles

If two adjacent angles of a four-sided figure are right angles, the area of the figure can easily be found without the use of surveyor's instruments or of higher mathematics.

Suppose that we desire to find the area in square feet of the figure *ABCD*. We have already learned how to find the area of the rectangle *ACED*, which is simply to multi-

ply the length by the breadth. Now, in the same way we can find the area of the smaller rectangle *EDBF* shown by the dotted line, and half of this will be the area of the triangle *EDB*, and the rectangle and triangle together make up the whole figure.

NOTE. — If *ACD* be not a right angle, the area of the figure can still be found, only we shall have two triangular areas to add to the rectangle instead of one. See Lesson No. 9.



EXERCISES

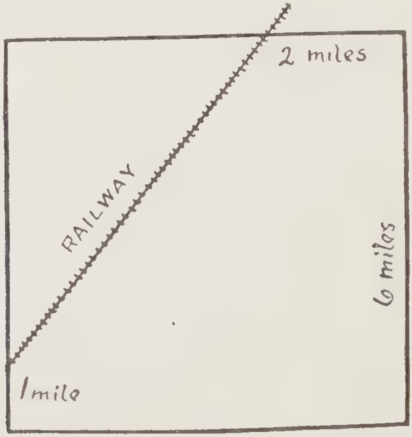
1. Find the area in feet of a four-sided figure having three right angles. The length of the longer side is 12 ft. and of the shorter side 9 ft. The breadth is 8 ft.

2. Find the cost of sodding a garden plot of the shape of the figure described above. *AB* is 60 ft.; *CD* is 45 ft.; *AC* is 36 ft. The price to be 20¢ a square yard.

3. A farm lies between two parallel roads with two corners right angles as in the above diagram. It has a frontage upon one road of $2\frac{1}{2}$ mi. and upon the other road of 2 mi. The distance between the roads is 2 mi. How many acres in the farm?

4. A city lot is surrounded by four streets, two of which are parallel and 20 rd. apart. It has a frontage upon one of these streets of 132 yd. and upon the other of 33 yd. A third street is at right angles to these two. Find the value of the lot at $ 800 an acre.

5. A railway crosses a Minnesota township as shown in the diagram. It enters one mile from one corner and leaves the township two miles from the opposite corner. How many acres on each side of the track?



Lesson No. 6. Measurements of Solids and Volumes

A rectangular **solid** is a body bounded by six rectangular surfaces. If all the sides are squares, the body is called a **cube**. If the sides are each a foot square, the body is called a **cubic foot**. The **volume** is expressed by the product of the length, breadth, and height. The three dimensions must be expressed in the same terms before multiplying — that is, they must be all in inches, or in feet, or in yards.

TABLE

27 cubic feet = one cubic yard.
1 cubic yard = one load.
1 cord = 128 cubic feet.

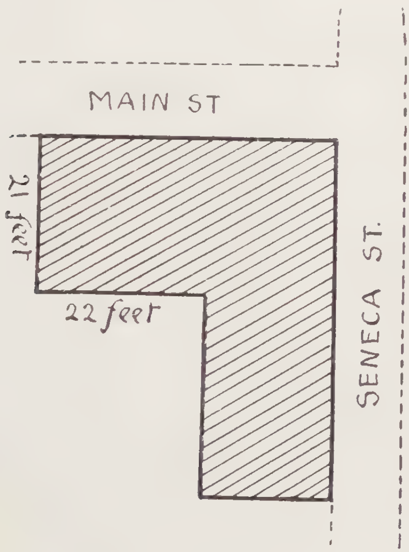
EXERCISES

1. What is the volume in cubic feet of a rectangular solid 3 yd. long, 2 ft. wide, and 6 in. high?

2. Find the cost of 32 blocks of marble each $2\frac{1}{2}$ ft. by 2 ft. by $1\frac{1}{2}$ ft. at $ 1.75 per cubic foot.

3. How many loads of gravel will be required for a road 3 mi. long if spread 9 ft. wide and 8 in. deep?

4. A pile of tanbark is 27 yd. long, 12 ft. high, and 16 ft. wide. What is it worth at $ 7.20 a cord?



5. A business block is to be erected with a frontage of 40 ft. on Main Street and 48 ft. upon Seneca Street. The other measurements are as shown in the diagram. Find the cost of excavation to a depth of 12 ft. at 30¢ a load.

Lesson No. 7. Estimates involving Solids and Volumes

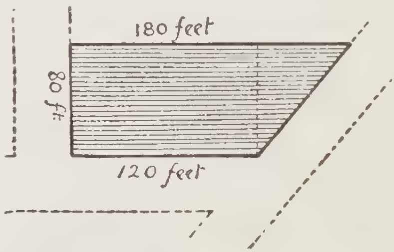
EXERCISES

1. Three-foot wood is piled 8 ft. high. How long should the pile be to contain 12 cords?

2. How many cut stones, each 9 in. long by 4 in. wide and 5 in. deep, will it take to pave a street 66 ft. wide and a half-mile long?

3. What weight of water will a rectangular tank contain, the length being 5 ft., the breadth 4 ft., and the depth 16 ft.? (A cubic foot of water weighs $62\frac{1}{2}$ lb.)

4. Find the cost of digging a ditch $2\frac{1}{2}$ mi. long by $1\frac{1}{2}$ ft. wide and 6 ft. deep at 35¢ a load.



5. An office building is to be of the dimensions shown in the above diagram. The cellar excavations are to be made to a depth of 15 ft. Find the cost at 27¢ a load. (Find the surface area as shown in Lesson 5; then multiply by the depth.)

Lesson No. 8. Lumber and Timber Measurements

A **foot of lumber**, as the term is generally used, means a piece of board one foot square and one inch thick, or of an equivalent number of cubic inches (144) of wood in some other shape. When lumber is so measured the result is frequently spoken of as "board measure," or B. M.

To find the number of feet of lumber in a board or plank, that is, the number of feet "board measure," multiply the length in feet by the width and thickness in inches, and divide the product by 12.

If the boards are 12 ft. long and 1 in. thick, the width in inches represents the number of board feet.

To find the number of feet of lumber in a floor, or sidewalk, or fence, or platform, or any rectangular surface, multiply the length in feet by the breadth in feet, and if more than 1 in. thick, multiply by the thickness in inches.

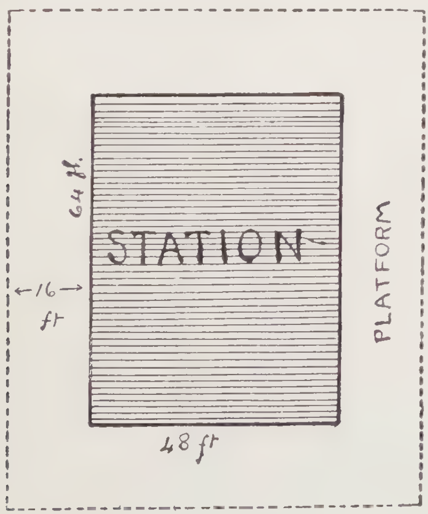
EXERCISES

1. How many feet of lumber in a plank 18 ft. long by 18 in. wide and 2 in. thick?

2. How many feet of lumber in a stick of timber 24 ft. long and 18 in. square?

3. Find the cost of the lumber necessary for a floor 20 ft. by 18 ft. at $ 25 per thousand feet.

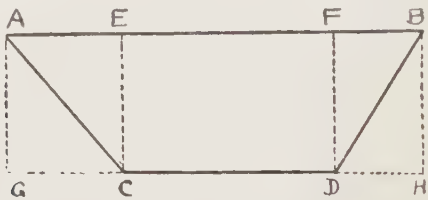
4. Find the value of 26,340 ft. of lumber at $ 27.50 per thousand.



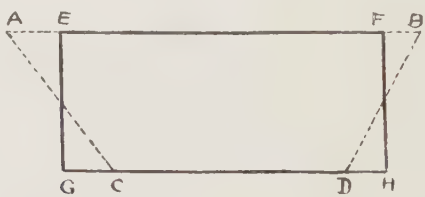
5. A railway station is 64 ft. by 48 ft. The platform surrounding it is 16 ft. wide and is built of 2-in. plank. Find the value of the lumber used at $ 30 per thousand feet.

Lesson No. 9. Areas of Trapezoids

If two sides of a four-sided figure are parallel, the area can easily be found if we know the perpendicular distance between the parallel sides. Four-sided figures that have two opposite sides parallel are called trapezoids.



Suppose that we desire to find the area in square feet of the figure *ABCD*. We form the three rectangles as shown, and then find the area separately of the rectangles. Only half in each instance of the rectangles *AGCE* and *FDHB* would be taken in finding the total area.



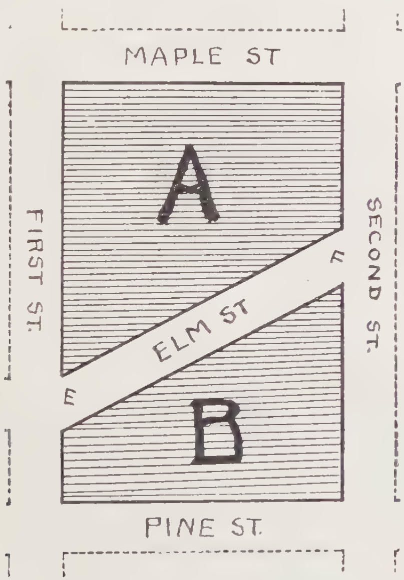
Notice that the figure *ACDB* is the same in area as the rectangle *EGHF*. To find the length *EF* we add the lengths of *AB* and *CD* together and take half their sum. *EF* is the *average* length of *AB* and *CD*.

Now to find the area of a four-sided figure two sides of which are parallel, we multiply the average length of the parallel sides by the perpendicular distance between them.

EXERCISES

1. Find the area of a four-sided figure having two sides parallel. The lengths of the parallel sides are 12 ft. and 9 ft., respectively, and the perpendicular distance between them is 14 ft.

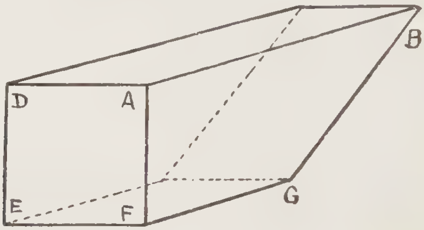
2. Find the cost of an inlaid wood floor the shape of the figure described above. *AB* is 28 ft. and *CD* is 18 ft. The floor is 15 ft. wide. The price is to be 18¢ a square foot.



3. Two city blocks are of the shape *A* and *B* shown in the figure. *A* has a frontage of 140 yd. on First Street, and 60 yd. on Second Street. *B* has a frontage of 24 yd. on First Street, 104 yd. on Second Street. The distance between First Street and Second Street is 272 ft. Find the total value of the land in *A* and *B* at 3¢ a square foot.

4. Find the cost of paving Elm Street from *E* to *F*, the street being 60 ft. wide, measured parallel with First or

Second Street, at a cost of \$1.95 per square yard. (You know the length of the parallel ends and the perpendicular distance between them.)



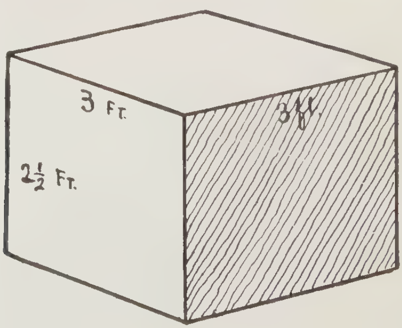
5. A piece of marble is of the above shape. The end is a square, each edge of which is 2 ft. AB is $5\frac{1}{2}$ ft. and FG is 3 ft. What is the piece worth at \$2 per cubic foot? (Find the area in square feet of $AFGB$ and multiply by the thickness from A to D to get the number of cubic feet.)

Lesson No. 10. Lumber and Timber Estimates

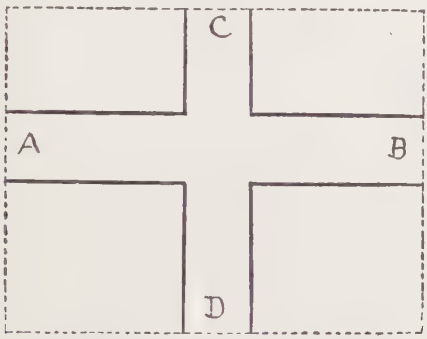
EXERCISES

1. Find the value at \$20 per thousand feet of the lumber necessary to build a tight board fence 12 ft. high around an athletic field 40 rd. square.

2. A plank sidewalk 18 ft. wide and 2 in. thick is to be built around a block 5 ch. square. What will the lumber cost at \$30 per thousand?



3. A packing-box manufacturer gives an estimate of \$424.20 for 300 boxes made of inch lumber, each box being $2\frac{1}{2}$ feet by 3 feet by 3 feet. The boxes cost 20¢ each to make, and the lumber is worth \$18 per thousand. Allowing \$6 for incidental expenses, what is his profit on the job?



4. An office building has 10 hall floors of the shape shown in the above diagram. From *A* to *B* is 80 ft., and from *C* to *D* is 60 ft. The halls are $7\frac{1}{2}$ ft. wide. Find the cost of the hardwood floors for these halls at an expense of \$60 per thousand, inch lumber.

5. A bridge 132 yd. long and 18 ft. wide is covered with plank $2\frac{1}{2}$ in. thick. Find the value of the plank at \$25 per thousand.

Lesson No. 11. Review

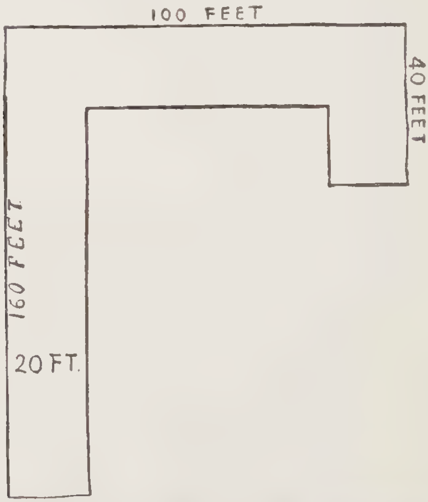
EXERCISES

1. How many sods, each 2 ft. 6 in. by 10 in., will be required to sod an acre of ground?

2. How many lots of one-quarter acre each can be made out of a piece of village property 45 ch. square?

3. What will it cost to paint the walls and ceiling of a hall 48 ft. long, 27 ft. wide, and 18 ft. high, at \$1.10 per square yard.

4. What will it cost to slate a roof, 44 ft. by 35 ft., at \$ 5.75 per square of 100 sq. ft.?



5. Find the value of the 4-in. plank necessary to cover a pier of the shape and length shown in the diagram, at \$ 40 per thousand feet.

Lesson No. 12. Plastering, Painting, and Calcimining

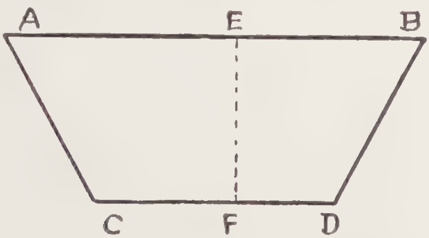
Plastering, painting, and calcimining are usually computed by the square yard. Allowances are sometimes made for openings in the walls such as doors and windows, but custom varies as to how much allowance should be made. Often one-half the area of the openings is deducted. In large contracts it might be well to have a written agreement regarding this matter, to avoid complications at the time of settlement.

EXERCISES

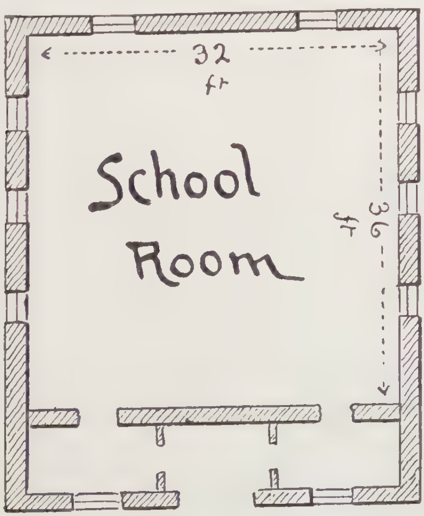
1. An assembly hall is 64 ft. long, 30 ft. wide, and 20 ft. high. Find the cost of plastering the walls and ceilings at

11¢ a square yard, allowing one-half rate for ten windows each 12 ft. by 4 ft.

2. What will it cost to paint a floor, 22 ft. long by 18 ft. wide, at 8¢ a square yard?



3. The floor of a room is of the shape shown in the diagram. AB is 30 ft., CD is 20 ft., and EF is 18 ft. The ceiling is 16 ft. high. Find the cost of painting the floor and ceiling at 12¢ a yard.



4. The above plan shows the rooms of a country schoolhouse. The dimensions of the main room are 32 ft. by

36 ft., and the height of the ceiling 13 ft. The doors are 4 ft. by 8 ft., and the windows 3 ft. by 6 ft. There is a baseboard around the room 1 ft. high. Find the cost of plastering the main room, walls, and ceiling at $13\frac{1}{2}$¢ a square yard, allowing one-half for doors and windows, and also allowing the full depth of the baseboard. (The doors are only 7 ft. above the baseboard.)

5. Find the cost of a piece of oilcloth, $18\frac{1}{2}$ ft. by 16 ft., at 80¢ a square yard.

Lesson No. 13. Measuring Stone Work

The methods of measuring differ greatly in different localities. The most common method of measuring stone work is by the **perch**. A perch of stone masonry consists of $24\frac{3}{4}$ cu. ft. For the foundation it is customary to take the outside measurements of a wall. This principle applies to all kinds of estimating.

RULE: *To find the number of perches in a stone wall multiply the number of cubic feet by* 4 *and divide by* 100; *then add one per cent; that is, one one-hundredth.*

NOTE. — This rule though not exactly accurate is accurate enough for all practical purposes. To find the accurate measurement one should divide the number of cubic feet by $24\frac{3}{4}$. This may be done by multiplying by 4 and dividing by 99.

Suppose that it is necessary to find the number of perches in a stone wall 48 ft. long, 3 ft. high, and $1\frac{1}{2}$ ft. thick.

$$48 \times 3 \times 1\tfrac{1}{2} = 216 \text{ cu. ft.}$$

$$\begin{array}{rl} 216 & \\ 4 & \\ \hline 8.64 & \text{Cut off two decimal places.} \\ .0864 & = \text{one per cent.} \\ \hline 8.7264 & = \text{number of perches.} \end{array}$$

The answer is 8 perches and a fraction, or, in actual practice, 9 perches. If we divide the number of cubic feet by $24\frac{3}{4}$, the same result is found.

Do the exercises which follow and give the results in *even* perches? No matter how small the fraction of a perch, call it one perch.

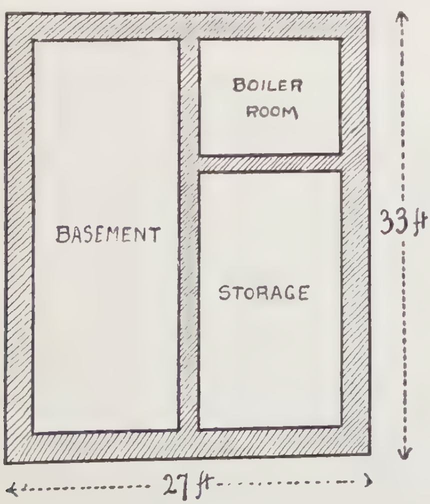
EXERCISES

1. Find the number of perches in a stone wall 99 yd. long, 4 ft. high, and 2 ft. thick.

2. Find the number of perches in a rectangular foundation 33 ft. by 22 ft., outside measurement, 12 ft. high and 2 ft. thick.

NOTE. — In questions of this sort consider the outside measurements of the wall, both in length and breadth, as its true measurements for purposes of estimating.

3. Find the cost of a stone wall one-eighth of a mile long, $3\frac{1}{2}$ ft. high, and $1\frac{1}{2}$ ft. thick, at $4.20 a perch.



4. Foundation walls are to be built according to the above ground plan. The outside measurement is 27 ft. by 33 ft. The main wall is 2 ft. thick; the wall through the center is $1\frac{1}{2}$ ft. thick, as is also the short cross wall. The depth of the foundation walls is 9 ft. Find the total number of perches of stone necessary.

NOTE. — In estimating the interior walls consider their full length; that is, outside measurement. For instance, the long wall through the center is 33 ft. rather than 29 ft. long.

5. A stone wall 14 ft. high is to be built around the grounds of a county jail. The shape is rectangular and the outside measurements are 198 yd. by 110 yd. The wall is 3 ft. thick at the base and 1 ft. thick at the top. Find the number of perches of stone necessary.

NOTE. — The average thickness is $(3 + 1) \div 2$ or 2 ft. To find the average thickness in any instance add the greatest thickness to the least and divide by 2.

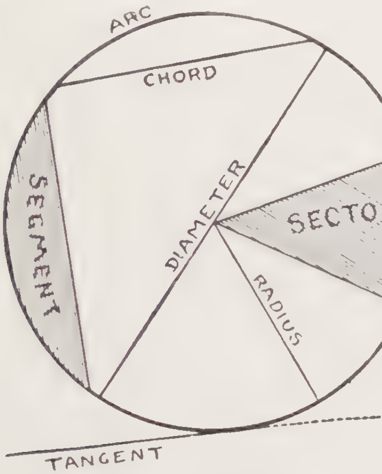
Lesson No. 14. Circumferences of Circles

The boundary line of a **circle** is called the **circumference.** Every point of the circumference is equally distant from the **center.** An **arc** is any part of the circumference. A **chord** is a straight line joining the extremities of an arc. A **radius** is a line drawn from the center to the circumference. A **diameter** is a chord passing through the center. A **tangent** is a line touching but not intersecting the circumference. A **segment** is the part of a circle cut off by a chord. A **sector** is the part of a circle included between two radii and the intercepted arc. **Concentric** circles have the same center. Note the following figure.

If the circumference of any circle were measured, its length would be found to be nearly $3\frac{1}{7}$ times the diameter. The real value of the ratio cannot be expressed in figures. To seven decimal places its value is 3.1415926. For all

ordinary purposes it is sufficiently accurate to say that the circumference is $3\frac{1}{7}$ times the diameter.

RULE: *To find the circumference of a circle multiply the diameter by* $3\frac{1}{7}$.



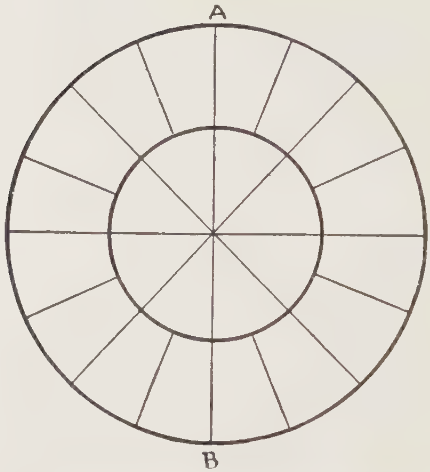
EXERCISES

1. Find the length in yards of the circumference of a circle whose diameter is 14 rd.

2. Find the entire length in miles of the drives of a park laid out in circular form, as shown in the diagram. The diameter AB is 7 mi. long, and the inner circular drive is half way between the center and the outer drive.

3. An athletic field is built in the form of a circle 210 yd. in diameter It is surrounded by a tight board fence 10 ft. high. Find the cost of the inch lumber necessary to build the fence, at $18 per thousand.

4. There are 12 pillars in a central court to be covered with gold leaf. They are 14 in. in diameter and 15 ft. high. What will it cost to put on the leaf at $37\frac{1}{2}$¢ a square foot?



5. What length of iron bar is necessary for four iron hoops to be put around a circular water tank $10\frac{1}{2}$ ft. in diameter, allowing 4 ft. extra for the welding of all four hoops?

Lesson No. 15. Areas of Circles

Rule: *To find the area of a circle, square the radius and multiply by* $3\frac{1}{7}$.

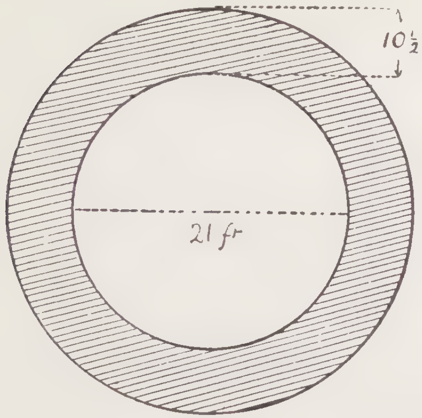
ILLUSTRATIVE EXERCISE

Find the area of a circle whose diameter is 14 in. The radius is 7 in.; $7 \times 7 \times 3\frac{1}{7} = 154$. The area is 154 sq. in.

EXERCISES

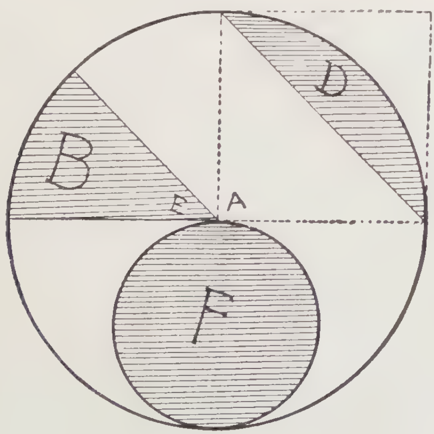
1. Find the area of a circle the diameter of which is 42 in.

2. The following diagram represents a circular walk, 10 ft. 6 in. wide, surrounding a monument 21 ft. in diameter. Find the area of the walk in square yards, and the cost of asphalt paving for it at \$2.70 a square yard.



3. A band stand is to be built in the form of a circle 28 ft. in diameter. Find the value of the 2-in. plank necessary for the floor at $22 per thousand.

4. A race course is laid out in the form of a circle. Give the diameter in yards that the track may be one mile.



5. The diameter of a circle as shown in the above figure is 56 in. The angle at A is a right angle and the angle at E is one-half a right angle:

(a) Find the area of the sector B.

(b) Find the area of the segment D.

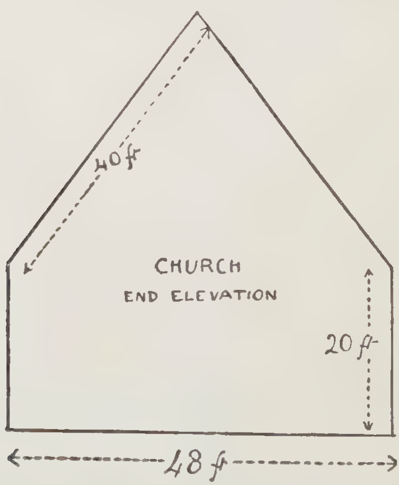
(c) Find the area of the circle F.

Lesson No. 16. Lathing and Plastering

Lathing is estimated by the square yard. Fifteen lath are counted to the yard, and $6\frac{1}{2}$ lb. of nails are allowed for each thousand lath. The lathing and plastering are usually estimated together. In the examples which follow no deductions are to be made for openings.

EXERCISES

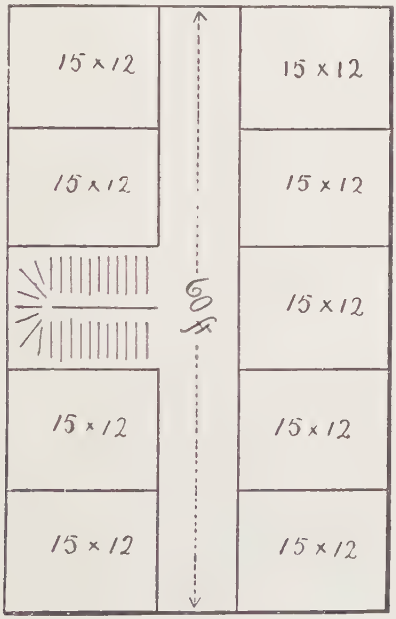
1. A room is 18 ft. by 16 ft., and 12 ft. high. Find the cost of lathing and plastering it at 21¢ a yard.



2. How many thousand lath will be needed for the walls and double ceiling of a church 60 ft. long and with end elevation as shown in the above figure?

3. Find the cost of a single coat of plaster on the walls and ceiling of a circular court-room the diameter of which is 35 ft., and the height of ceiling 27 ft., at 14¢ a square yard.

4. How many lath will be necessary for the court-room described in Exercise No. 3?



5. The above diagram represents a floor of an office building. The offices are 15 ft. by 12 ft.; the hall is 60 ft. long and $7\frac{1}{2}$ ft. wide; the ceilings are 12 ft. high; there are 6 floors. Find the cost of lathing and plastering throughout, including material and labor, of the 9 offices and the hall on each floor at 24¢ a square yard. (Omit in the estimate the shaded section representing the stairway.)

The four lessons here following are intended to review the work already covered. The student of this course who succeeds in solving correctly from fifteen to eighteen of these twenty review exercises may feel satisfied that he is fairly familiar with the work taken up so far.

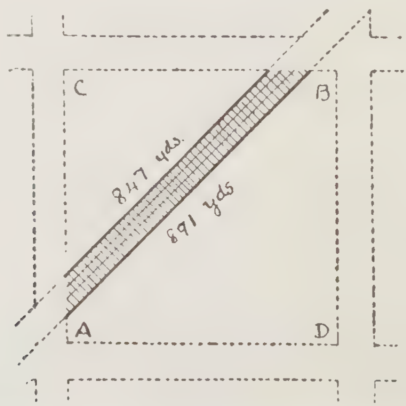
Lesson No. 17. General Review

EXERCISES

1. How many yards of fencing-wire will be required to inclose a 10-acre field 88 yd. wide, the fence to be made 5 wires high?

2. If railway ties are 3 ft. apart from center to center, how many ties are necessary for a road 12 mi. long?

3. A city council decides to spend \$25,000 in new sidewalks. The contractor gives an estimate of \$2.10 per square yard for asphalt. If the walks are to be $4\frac{1}{2}$ ft. wide, how many miles of sidewalks can they build? (Give the exact answer.)



4. A railway crosses a city block as shown in the diagram. The road is 66 ft. wide from fence to fence. The shorter fence measures 847 yd., and the longer fence 891 yd. A

board of arbitrators values the land at $8000 an acre. What will it cost the railway company?

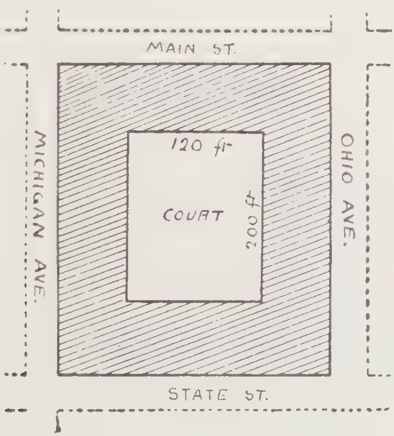
5. A pile of tanbark is 36 yd. long, 12 ft. high, and 18 ft. wide. What is it worth at $8.40 a cord?

Lesson No. 18. General Review (Continued)

EXERCISES

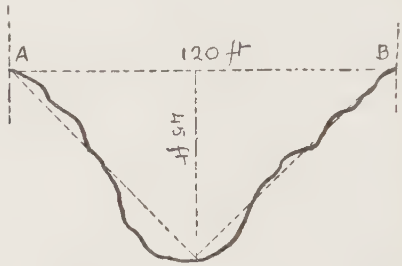
1. How many cut stones, each 9 in. long by 4 in. wide and 5 in. deep, will it take to pave a street 33 ft. wide from curb to curb, and a quarter of a mile long?

2. How many feet of lumber in 500 scantling, each 16 ft. long and $3\frac{1}{2}$ in. by 6 in.?



3. An office building is to cover an entire square. The frontage on Main and State streets will be 240 ft. each, and on Michigan and Ohio avenues 320 ft. each. The inner court will be 200 ft. by 120 ft. Find the cost of the excavations under the entire building to a depth of 12 ft. at 30¢ a load.

4. Find the cost of asphalting the court in problem No. 3 at $2.40 a square yard.



5. A small gully crosses some city property. It is 440 yd. long, and the lengths of a cross-section are as shown in the diagram; that is, 120 ft. from A to B, and 45 ft. deep. How many loads of earth will fill the gully?

Lesson No. 19. General Review (Continued)

EXERCISES

1. Find the cost of the inch lumber necessary for a tight board fence, 10 ft. high, around a circular field 280 yd. in diameter, at $15 per thousand.

2. The 5-ft. fences of a stock yard are as shown in the diagram. The divisions are even eighths, and halves and quarters of eighths. The total length is 108 rd., and the total breadth 72 rd. Find the cost of whitewashing all fences, both sides, at 2¢ a square yard.

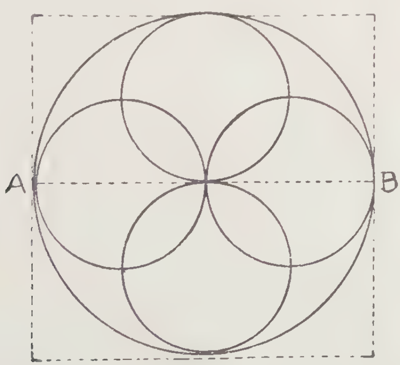
3. How many feet of inch lumber will it take to make 50 packing boxes, each 2 ft. by 3 ft. by $2\frac{1}{2}$ ft.?

Note. — In examples of this kind take outside measurement. The waste in sawing and from other causes would more than offset measuring the edges twice.

4. A soap manufacturer desires to pack 6 doz. cakes of soap in each box. The cakes when wrapped are $3\frac{1}{2}$ in. long, $2\frac{1}{2}$ in. wide, and $1\frac{1}{2}$ in. thick, and are to be laid flat and not

edgeways. Give the inside dimensions of a box which will serve for packing.

5. The five circular ridges of a ceiling, as shown in the diagram, are to be gilded at an expense of 20¢ a foot. If AB is 42 ft., find the entire cost.



Lesson No. 20. General Review (Continued)

EXERCISES

1. Find the number of perches of stone masonry in a rectangular foundation 36 ft. by 18 ft., outside measurement; the wall to be 12 ft. high and $1\frac{1}{2}$ ft. thick.

2. Twenty-four circular tanks, each 7 ft. in diameter and 8 ft. deep, but without covers, are to be painted inside and outside. Find the cost at $5\frac{1}{2}$¢ a square yard.

3. A circular lake, the radius of which is 63 rd., is frozen over to a depth of $1\frac{1}{2}$ ft. How many tons of ice can be taken from it if a cubic foot of ice weighs $57\frac{1}{2}$ lb.? (Take 2000 lb. to the ton. Multiply the area in feet by $1\frac{1}{2}$ to get the number of cubic feet.)

4. Find the cost of a single coat of plaster on the walls and ceiling of a room 24 ft. by 18 ft., and 10 ft. high, allowing 100 ft. for doors, windows, etc., at 15¢ a square yard.

5. How many feet of lumber in a circular platform 2 in. thick and 28 ft. in diameter?

6. The pickets of a fence are 3 in. wide and 2 in. apart. How many pickets in a fence one-fourth of a mile long?

7. A surveyor reports the length of a projected railway as 19,680 ch. What will it cost to build the road at an estimate of $ 18,000 per mile?

8. Find the cost of 24 blocks of marble, each $1\frac{1}{2}$ ft. by 1 ft. by 10 in., at $ 1.50 a cubic foot.

9. How many loads of gravel will be required for a road 10 mi. long, if spread 9 ft. wide and 3 in. deep?

10. A hundred telegraph posts, each 14 in. in diameter, are to be painted to a height of $7\frac{1}{2}$ ft. Find the cost of the painting at 9¢ a square yard.

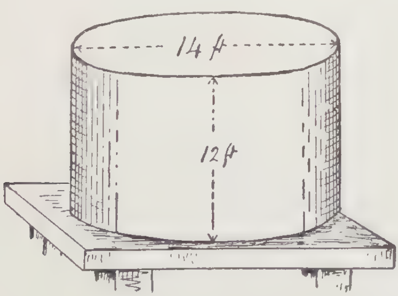
Lesson No. 21. Volumes and Weights of Liquids

There are about 231 cu. in. in a gallon. It is often necessary to estimate the number of gallons of water or other liquid in a tank, or to get the dimensions of a tank required to contain a stipulated number of gallons. The weight of water in a tank can readily be found when we know that a cubic foot of water weighs $62\frac{1}{2}$ lb.

EXERCISES

1. A rectangular reservoir is 22 ft. long, 21 ft. wide, and 6 ft. deep. How many gallons of water will it contain?

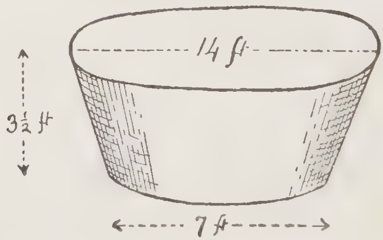
2. A water tank contains 17,280 gal. of water. What weight in pounds does it contain?



3. A circular water tank is of the dimensions shown in the diagram. (*a*) What weight of water does it contain when full? (*b*) How many gallons of water does it contain when full?

NOTE. — To find the volume of a circular solid similar to the above, find the area of one end and multiply by the depth.

4. A rectangular trough is to be built to hold 216 gal. It is to be 11 ft. long and 18 in. wide, inside measurement. How deep should it be?



5. There are three vats in a manufacturing establishment, each of the shape and dimensions shown in the diagram. They are each half full of a liquid dye. How many gallons of the dye in the three vats? (See "Note" in the "Answer.")

Lesson No. 22. Lumber and Timber Estimates (Continued)

In measuring common lumber it is customary to consider lumber less than an inch in thickness as **inch** lumber. The more expensive lumber used in furniture and for finishing are charged for by the foot, or by the hundred feet, rather than by the thousand feet.

EXERCISES

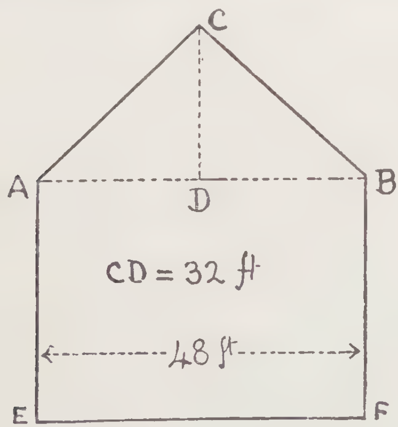
1. How many inches in width must a lumber dealer measure off in measuring 14-ft. lumber $1\frac{1}{2}$ in. thick, to make 2170 ft.?

2. Find the value of the inch lumber in a sidewalk 6 ft. wide around a square 60 yd. by 40 yd.; and two walks, each 4 ft. wide, through the middle, at right angles to each other and parallel to the sides, at $ 24 a thousand.

3. A stick of timber 2 ft. square and 16 ft. long is sawed into inch lumber. How many feet will it make, one-sixth being lost in sawing?

4. A board fence with posts 7 ft. apart is built with one 10-in., one 8-in., and three 6-in. boards. Find the value of sufficient 14-ft. lumber to build 40 rd. of the fence at $ 22 per thousand.

5. Find the value of the $1\frac{1}{2}$ in. plank, at $ 24 per thousand, that will be needed to make a land roller whose diameter is $3\frac{1}{2}$ ft. and length 9 ft.

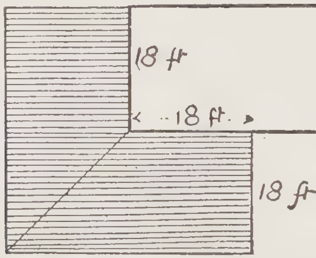


6. How many feet of inch lumber in the two ends of a barn the elevation of which is shown in the accompanying diagram? *AE* or *BF* is 36 ft.

7. How high should the ceiling of a schoolroom 20 ft. by 25 ft. be, to allow 150 cu. ft. of air to each of 50 pupils?

8. If a bushel of grain occupies $1\frac{1}{4}$ cu. ft., how deep must the grain be in an 18-ft. by 5-ft. bin, to contain 800 bu.?

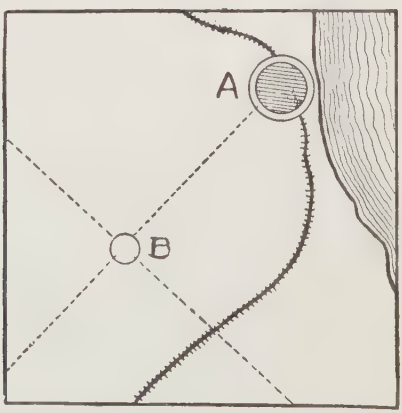
9. What will it cost to tuckpoint one side of a brick wall, 120 ft. long by 12 ft. high, at 70¢ a square yard?



10. Find the value of the hardwood inch lumber in the floor of veranda, as shown in the shaded section of the drawing, at $ 40 per thousand.

Lesson No. 23. Maps and Plans

Nearly all maps or plans are drawn to a certain **scale.** For instance, a scale of three miles to an inch in a map would indicate that two points upon the map an inch apart are actually three miles apart.



Suppose that the map shown here is drawn on a scale of two miles to an inch, then the distance from A to B in

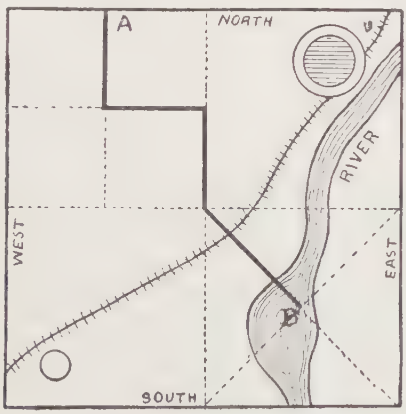
direct line, supposing that they are by actual measurement just one inch apart, is two miles.

EXERCISES

1. A map of an eastern township is drawn on a scale of 6 in. to the mile. The map is $3\frac{1}{2}$ ft. by 2 ft. How many acres are there in the township?

2. If the scale of a large map is 4 mi. to the inch, how many acres are there in a township which is represented by a square whose side is $1\frac{3}{4}$ in.?

3. A ground plan of a church of rectangular shape is drawn upon a scale of 3 ft. to a half inch. The drawing measures $10\frac{1}{2}$ in. by $7\frac{1}{2}$ in. What are the length and breadth of the building?



4. A map is drawn, as shown in the accompanying diagram, upon a scale of 1 in. to a mile. A trunk sewer is marked to enter at *A* and to cross the map as shown to a point at *B*, equidistant from the east and south sides, and three-fourths of the diagonal distance from the opposite corner. If the map is 2 in. square, what is the length in yards of the sewer?

5. A section of country is 90 mi. in greatest length, and 40 mi. in greatest width. A map is to be made for a book the leaves of which, exclusive of margin, are $4\frac{1}{2}$ in. by 3 in. Upon what scale should the map be drawn so that it may be as large as the book's size will permit?

Lesson No. 24. Shingles — and Lathing and Plastering (Continued)

One thousand *shingles* laid 4 in. to the weather will cover 100 sq. ft. of surface; and 5 lb. of shingle nails will fasten them on. There are usually 250 shingles to a bunch.

Eight bushels of good lime, 16 bu. of sand, and 1 bu. of hair will make enough good mortar to plaster 100 sq. yd.

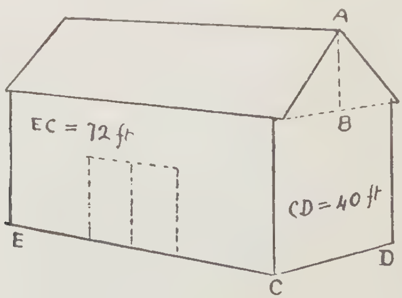
The common laths will lay 48 ft. to the bundle.

EXERCISES

1. How many shingles are required for a roof with an area of 1250 sq. ft.?

2. How many bunches of shingles are required for a double roof 24 ft. long with 20-ft. rafters? (Give full bunches. Dealers do not break bunches.)

3. A plasterer makes a bid of $62.50 on a job of plastering the four sides of a hall 42 ft. by 24 ft., and 18 ft. high. Lime costs him 50¢ a bushel, hair $1.50 a bushel, and sand 4¢ a bushel. He pays 17¢ a yard for labor. Will he make or lose on the job, and how much?



4. A barn as shown in the plan is 72 ft. by 40 ft. From A to B is 15 ft. How many bunches of shingles laid 4 in. to the weather will be necessary to cover the double roof?

5. In the fourth example how many feet of inch lumber will be necessary to cover the sides and ends? The height of the barn from eaves to foundation is 30 ft.

6. Find the cost of plastering the walls of a room 18 ft. by 24 ft., and 14 ft. high, at 25¢ a yard?

7. Find the cost of cementing the sides and bottom of a cistern 8 ft. by 12 ft., and 9 ft. high, at 8¢ a square foot.

8. How many pounds of paint will cover the front of a brick house 30 ft. wide and 27 ft. high, allowing for 6 openings 8 ft. by $3\frac{1}{2}$ ft., if 1 lb. covers $2\frac{2}{3}$ yd. of bricks?

9. Two strips of molding at $37\frac{1}{2}$¢ a foot are placed around a drawing-room 24 ft. by 16 ft. Find the cost.

10. Find the cost of the blinds of 8 store windows, each 6 ft. wide by $7\frac{1}{2}$ ft. high, at 35¢ a square yard.

Lesson No. 25. Studding, Joists, and Rafters

The exercises of this lesson involve a consideration of labor. To give a correct estimate upon a contract it is necessary to know:

(1) The exact measurements.

(2) The amount of material necessary.

(3) The cost of the material.

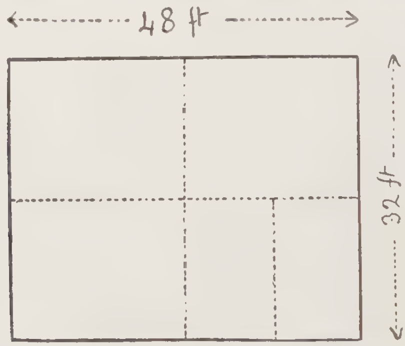
(4) The amount of work of particular kinds which one man can do in a day.

(5) The wages of men for special kinds of work.

The most accurate and the most valuable knowledge of this character is learned from actual experience; but, even though this knowledge is known, the process of putting it together to arrive at fair and just estimates involves such a knowledge of arithmetic and mathematics as it is the purpose of these lessons to teach.

EXERCISES

1. Studding is usually placed 16 in. from center to center. Two men will frame and place in a wood building not exceeding three stories, 700 lineal (*long*) feet of ordinary studding in one day. Find the cost of labor at $2.25 a day



per man for framing and placing studding in a two-story frame building, including partitions, as shown in the diagram, both floors being the same, and the height of each floor being 14 ft. (Consider only one studding at each corner.)

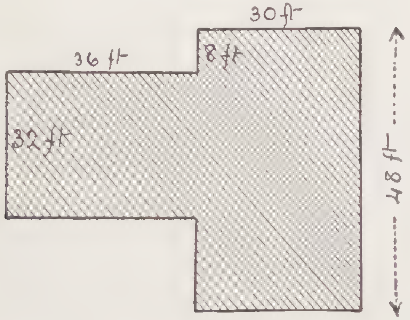
2. Joists are ordinarily placed 16 in. from center to center. Two men will frame and place in a wood building not exceeding three stories, 600 lineal (*long*) feet of ordinary joists in one day. Find the cost of labor at $2.25 a day per man for framing and placing joists in the two floors of the building described in exercise No. 1.

NOTE. — The outside joists in No. 2 and the corner studding in No. 1 will be slightly closer than 16 in., depending upon the thickness. Consider even 16 in.

3. Two men will in one day frame and place in a building 600 lineal feet of rafters — roof with plain gable. A particular building requires 2100 ft. of rafters. Find the cost of labor at $2.40 a day per man.

NOTE.—The contract price for framing one and a half, two, and two and a half story houses is very often based upon a certain price per 100 lineal feet of all the billed timber.

4. Suppose that for the framework of a building 34,000 lineal feet of studding, joists, etc., are required. The estimate for labor is 65¢ per 100 lineal feet. At $2.60 a day, how many days' labor will be required?



5. The tin roof of an office building is in measurements as shown in the diagram. A sheet of roofing tin is 14 by 20 in., and a box of tin contains 112 sheets and costs $6. Allowing for side ribs and laps, a box of tin will cover 180 sq. ft. It will take 10 lb. of solder to a box. One man will lay a box in one and a half days. Consider solder worth 15¢ a pound, and labor $2.40 a day. Find the cost of the roof.

Lesson No. 26. Brick Work

Bricks vary in size. The common brick is 8 in. by 4 in. by $2\frac{1}{2}$ in. The labor and material of brick work are usually estimated by the 1000 brick. In measuring up brick walls it is not a general custom to deduct for openings.

To ascertain the number of bricks in a wall first obtain the number of superficial feet; that is, the area of the wall; multiply this by 7 for a 4-in. wall, by 14 for an 8-in. wall,

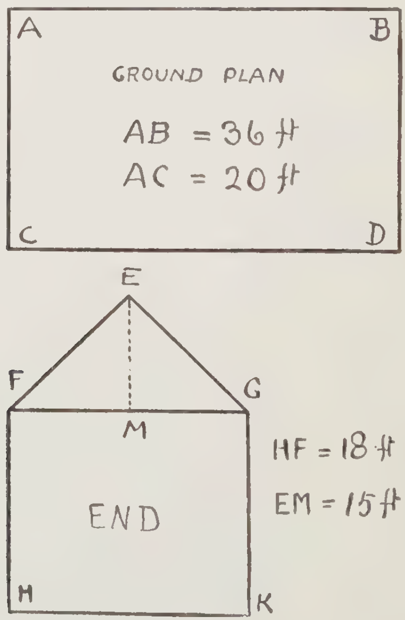
by 21 for a 12-in. wall, and so on. In ordinary shell walls take outside surface measurement to allow for the extra bricks needed in matching.

An average day's work is 1500 bricks. The number of bricks a mason will lay in a day on a plain wall will depend largely upon its thickness. The narrower the wall, the slower the work. Veneered work is much slower, from 400 to 600 bricks being regarded as a day's work.

EXERCISES

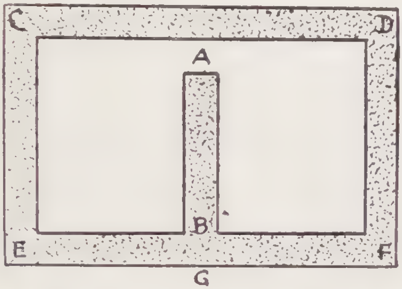
1. How many loose bricks in a pile of brick 12 ft. long, 8 ft. 8 in. wide, and 5 ft. high?

NOTE. — There is no mortar consideration, therefore find exact number of cubic inches in the pile, and divide by the exact number of cubic inches in a brick.



2. How many bricks will be needed to veneer a block-shaped house with dimensions as shown in the plans?

3. What will it cost to lay 48,000 bricks, including cost of bricks at \$6 per 1000, 3 barrels of lime at \$1.25 a barrel, 36 bushels of sand at 5¢ a bushel, and labor at \$2.60 a day of a man who can lay 1200 bricks a day, with helper at \$1.25 a day?



4. A solid brick wall 2 ft. thick and 15 ft. high, with ground plan as shown in diagram, is to be built. *CD* is 180 ft.; *CE* is 160 ft.; and the opening at *A* is 12 ft. Consider the inside wall as length *AB* and not length *AG*. Find the entire cost of the work upon the following basis:

(1) Bricks to cost \$5.40 per 1000.
(2) Lime to cost \$1 per barrel.
(3) One barrel of lime to 2000 bricks.
(4) Sand to cost 5¢ a bushel.
(5) Seven bushels sand to 1000 bricks.
(6) Laying bricks \$2.80 per 1000.

5. Find the cost of tuckpointing the brick walls described in Exercise No. 4, inside and outside, at 60¢ a square yard.

Lesson No. 27. Review

EXERCISES

1. Find the area in acres of a square field whose side is 396 ft.

2. How many common bricks laid flat will be required for a sidewalk 7 ft. 4 in. wide by 200 ft. long?

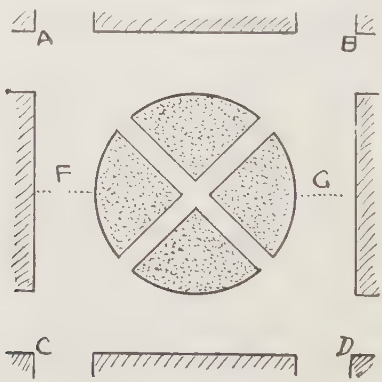
3. A rectangular garden, 64 ft. long and 35 ft. wide, is surrounded by a walk 3 ft. wide. Find the cost of the walk at 21 ¢ a square foot.

4. A map is 1 ft. 4 in. long and 1 ft. wide. If the scale of the map is $2\frac{1}{2}$ mi. to an inch, how many square miles of country does it represent?

5. The dimensions of a rectangular floor are 22 ft. 6 in. by 16 ft. 9 in. What will it cost to cover it with oilcloth at $ 1.52 a square yard?

6. How many paving stones, each 7 in. long and 4 in. wide, will be required to pave one-quarter of a mile of street, 63 ft. wide?

7. A circular pond, 140 ft. in diameter, is surrounded by a walk 4 ft. wide. Find the area of the walk.



8. A city square is to be laid out as shown in the above diagram. The square *ACDB* from curb to curb is one-sixteenth of a mile on each side. The center is a circular grass plot, with two asphalt walks, each 6 ft. wide, and crossing in the center. From curb to circle at *F* and *G* and the other two sides is in each case 72 ft. The entire street·

inside of the described square exclusive of the circle, but including the two walks, is asphalted at a cost of $2.40 a square yard. Consider the length of the cross walks in each instance as the diameter of the circle, and find the entire expense.

Lesson No. 28. Square Root

It is necessary here that the student should understand how to find the square root of large numbers. The square of a number is the product of the number by itself; as, 25 is the square of 5, 16 is the square of 4. The square root of a number is the number which multiplied by itself will produce the number; as, 5 is the square root of 25, 12 is the square root of 144.

Suppose that it is necessary to find the square root of 55,225. The process is as follows:

```
2    | 5'52'25 | 235
     | 4
43   | 152
     | 129
465  |  2325
     |  2325
```

235 = the square root.

Explanation. — Point off the number into periods of two figures each, beginning at the right. Find the largest possible number which, when squared, will divide the first period. 3 squared is 9, and is too large. 2 squared is 4, and will divide the first period, which is 5. Place the 2 as divisor at the left and in the quotient space at the right. Divide, subtract, and bring down as in long division, except that you must bring down two figures (a period) instead of one figure. Now we

change the divisor each time. The new divisor is found by doubling the answer thus far secured. 2 doubled gives 4. 4 into 15 (omitting the 2) will go 3 times. Place the 3 in the answer space, and also after the 4 in the divisor. Now multiply. 3 times 43 equals 129. Subtract and bring down as before. For a new divisor double the 23, and we have 46. This divided into 232 (omitting the 5) will go 5 times. Place the 5 in the divisor and multiply. Then 235 is the required square root.

EXERCISES

Find the square root in each instance:

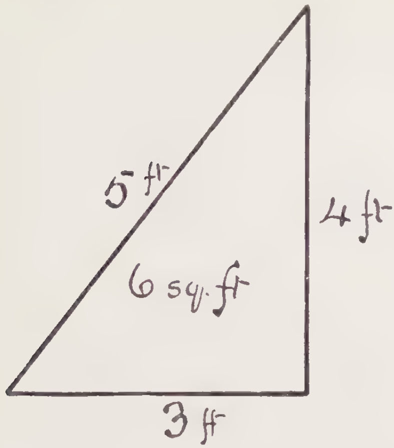
1. 625.	9. 21,025.	17. 5625.	24. 17,424.
2. 3025.	10. 77,284.	18. 4096.	25. 21,025.
3. 5476.	11. 196.	19. 5184.	26. 16,384.
4. 9604.	12. 225.	20. 6561.	27. 576.
5. 11,881.	13. 529.	21. 17,956.	28. 41,616.
6. 13,225.	14. 361.	22. 7056.	29. 11,025.
7. 41,616.	15. 729.	23. 11,664.	30. 2601.
8. 57,121.	16. 1024.		

Lesson No. 29. Areas of Triangles

In the measurement of triangular surfaces the areas can be found without the use of surveyors' instruments by the following rule:

To find the area of a triangle: From one-half the sum of the three sides subtract each side separately; multiply the three remainders and the half-sum together; find the square root of the product.

To prove that this rule is true, let us take the case of a right-angled triangle, the dimensions of which are as shown in the diagram.



Proceed to find the area according to the rule already stated.

Sum of three sides $= 3 + 4 + 5 = 12.$

One-half the sum $= 6.$

$$\begin{array}{ccc} 6 & 6 & 6 \\ 3 & 4 & 5 \\ \hline \end{array}$$
$$3 \times 2 \times 1 \times 6 = 36.$$

The product of the three remainders and the half-sum is 36.

The square root of 36 is 6.

Therefore, the area of the triangle as found by the rule is 6 sq. ft.

But we already know (see Lesson No. 5) that the area of a right-angled triangle is found by multiplying the base by the height and taking half the product; or, in this case,

$$3 \times 4 \div 2 = 6.$$

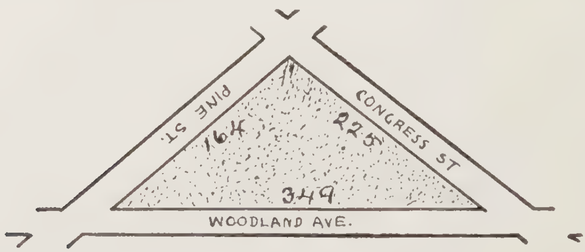
Therefore, the rule is true in the case of right-angled triangles. In trigonometry it is shown that it is true generally.

EXERCISES

1. Find the area of a triangle the sides of which are 29 yd., 52 yd., and 69 yd.

2. Find the area of a triangle the sides of which are 57 yd., 60 yd., and 111 yd.

3. Find the area of a triangle the sides of which are 143 yd., 296 yd., and 349 yd.



4. Three streets of a city form a triangular lot, as shown in the diagram. The frontage on Woodland Avenue is 349 yd., on Pine Street 164 yd., on Congress Street 225 yd. Find the area in square yards of the lot.

5. Find the area of a triangle the sides of which are 150 ft., 120 ft., and 145 ft.

Lesson No. 30. House-building

Experienced builders, in figuring upon the cost of a house or of a series of houses, sometimes base their estimate upon the number of square feet of floor surface, and sometimes upon the gross number of cubic feet in the rooms. Both methods are fairly reliable under certain conditions. For one style of finish one price per foot is charged, while for a

more expensive finish a higher price is charged. If there are items or fittings not usually included in ordinary priced city houses, these are estimated separately and are added to the general estimate. A builder having built a particular style of house and knowing the exact cost per square foot or per cubic foot can very easily apply the same estimate to other and similar houses, adding or subtracting, as the case may require, for dearer or cheaper finishings.

EXERCISES

1. A seven-roomed city house, two stories, 14 ft. frontage and 40 ft. deep, costs $ 784 to build. What is the rate per square foot of floor?

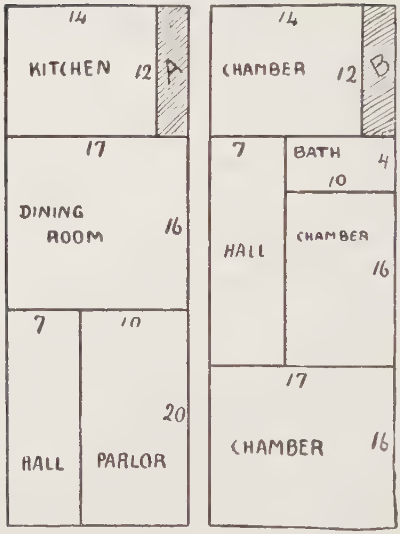
2. An eight-roomed city house, three stories, 16 ft. frontage, and 50 ft. deep, costs $ 2160 to build. What is the rate per square foot of floor?

3. On an estimate basis of 85¢ per square foot and 10% added for extras, find the cost of building a row of 15 houses, each of three stories, and each containing 864 sq. ft. to a floor.

The student will notice that an estimate based upon the floor area of a particular size of house cannot be applied without modification to a house twice as large or four times as large. The contractor usually has a price for each size and style — that is, in a general way — based upon figures actually obtained in the building of houses. In using the estimates of a house of one size in finding the cost of a house of a larger or smaller size, it is safer to figure on the cubic foot basis.

4. The measurements of a seven-roomed house, in feet, omitting the shaded sections *A* and *B*, are as follows:

(*a*) Upon a basis of 90¢ per square foot, find the cost of building 30 such houses.



(*b*) If the ceiling of each floor is 9 ft. from the floor, what estimate per cubic foot will equal 90¢ per square foot?

5. A contractor has built a house containing 120,500 cu. ft., and he finds that the actual cost was $13\frac{1}{2}$¢ per cubic foot. He is now figuring on a house containing 96,400 cu. ft. The material and labor will average him 10% more, and he wishes to make 20% profit upon his actual outlay. Give the face value of his bid.

Lesson No. 31. House-building (Continued)

EXERCISES

1. A 12-roomed city house, three stories, 18 ft. frontage and 42 ft. deep, costs 94¢ per square foot of floor. Find the gross cost of 200 such houses.

2. A contractor finds that a particular house that cost him $ 1191.36 has rooms as follows:

First floor . . .	Parlor	14 ft. by 16 ft.
	Hall	8 ft. by 16 ft.
	Dining room . . .	22 ft. by 14 ft.
	Kitchen	18 ft. by 12 ft.
Second floor . . .	Chamber	22 ft. by 12 ft.
	Chamber	18 ft. by 12 ft.
	Chamber	14 ft. by 10 ft.
	Bath	8 ft. by 14 ft.
	Upper Hall . . .	18 ft. by 8 ft.

With ceiling upon the first floor of 9 ft. and on the second floor of 8 ft.

He has another house, the gross cubical measurement of which is 12,410 cu. ft., upon which he is figuring. At the same rate per cubic foot, how much should his bid be to make $ 350 profit?

3. An office building has 10 floors, each 40 ft. by 72 ft., with 10 ft. ceilings throughout. The building cost $ 72,000. What is the rate per cubic foot?

4. What should be the cost of an office building of 9 floors, each 64 ft. by 36 ft. with 10 ft. ceilings, figured at the cubic rate upon the basis of the cost of the building described in No. 3?

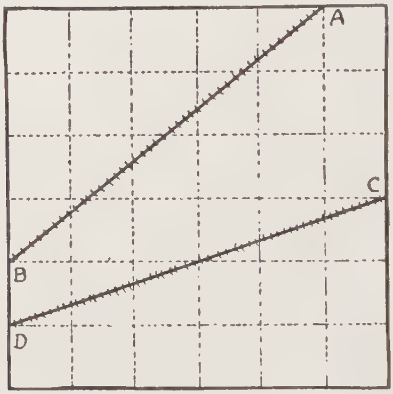
5. On an estimate basis of 9½¢ a cubic foot and 20% added basis, find the contract price for a house of 47,500 cu. ft.

Lesson No. 32. Review

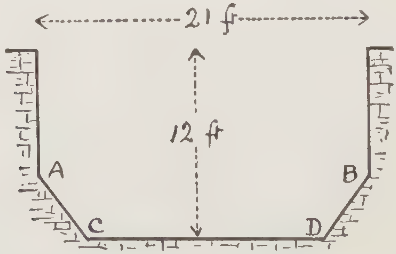
EXERCISES

1. Find the area of a 4-sided figure having 2 sides parallel. The lengths of the parallel sides are 18 ft. and 10 ft. respectively, and the perpendicular distance between them 24 ft.

2. A township is 6 mi. square. One railway crosses from A to B and another from C to D. Find the area of the land between the railway lines.



3. The cross-section of a canal is as shown in the diagram: 21 ft. wide, 12 ft. the greatest depth, 8 ft. deep to A or to B, with CD 15 ft. Find the cost of the excavation necessary per 100 lineal yards at 24¢ a load.



4. Find the value at $22 per thousand of the lumber necessary in building a tight board fence 13 ft. high around an athletic field 48 rd. square.

5. How many lots of $\frac{1}{4}$ acre each can be cut out of a piece of city property 360 rd. by 180 rd.?

Lesson No. 33. Painting, Paperhanging, Roofing, and Slating

In painting, the rule is to measure wherever the brush goes and to charge by the superficial yard, except where it becomes necessary to work to a line, as in the case of skirtings, to prevent the floor or wall from being soiled, technically termed "cut on both edges."

A great deal of work, such as cornices, window sills, balusters, brackets, etc., should be estimated separately.

Paperhanging is paid for by the number of pieces or rolls. Odd yards are charged as one piece. The pumicing and preparing of walls should be considered in making a bid upon a particular piece of work. Charge for hanging and borders at a fixed rate per dozen yards running.

Measure roof slating by the square of 100 superficial feet. In measuring allow 6 in. extra for connections where there will be cutting and waste.

Measure gravel roofing by the square of 100 ft. Consider in the estimate the number of plies of tarred felt and the quantity of pitch and gravel used to the square. Make no deductions for traps smaller than a yard square.

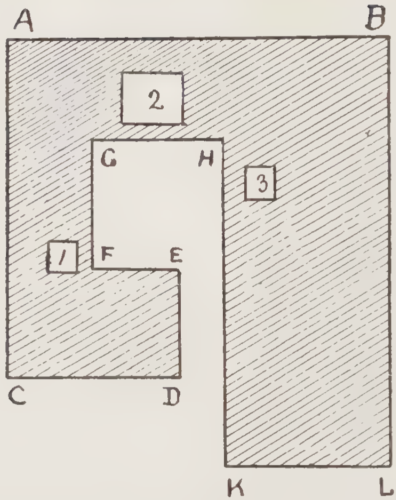
Gasfitters' and plumbers' work is usually estimated by the foot, running, according to the size and quality of pipe, with extras for elbows, crosses, sockets, etc.

EXERCISES

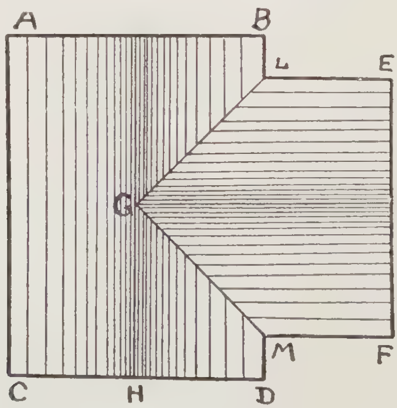
1. Find the cost of painting 294 doors (both sides) each 4 ft. 6 in. by 9 ft. 4 in., at $22\frac{1}{2}$¢ a square yard.

2. The flat roof of a city building is of the shape shown in the drawing and of the following measurements:

$AC = 64$ ft.	$BL = 80$ ft.
$HK = 60$ ft.	$GF = 24$ ft.
$ED = 20$ ft.	$FE = 16$ ft.
$GH = 24$ ft.	$KL = 32$ ft.
$AB = 72$ ft.	$CD = 32$ ft.

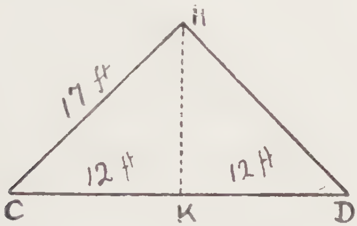


Find the cost of covering the roof with tarred felt and gravel at an expense of $ 9.36 per square of 100 ft., allowing for one opening, No. 2, which is 12 ft. on each side.



3. The roof of a dwelling house is of the shape shown in the drawing. The main building, *ACDB*, is 24 ft. by 32 ft.

The gables at AB, CD, and EF are of the dimensions shown in this drawing; that is, all three are the same size. The



right wing extends 15 ft. from the main building; that is, from BD. Allow 6 in. extra, 1 ft. in all, along each of the lines GL and GM for cutting and waste, and find the number of *squares* of slating necessary for the roof.

4. If wall paper is 18 in. wide and a single roll is 24 ft. long, how many rolls will be required for the 4 walls of a hall 48 ft. by 36 ft., and 18 ft. high?

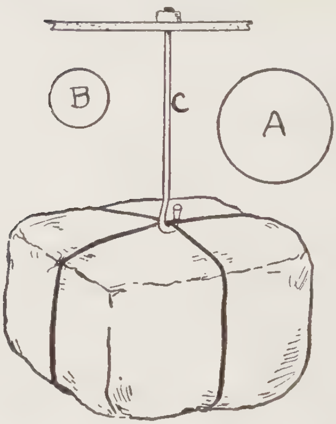
5. With paper 18 in. wide, at 40¢ a roll, what will it cost to paper the 4 walls and ceiling of a room 24 ft. by 36 ft., and 12 ft. high? Consider the rolls 24 ft. long.

Lesson No. 34. Strength of Rods

By the **tenacity** of a body is meant its strength to resist **tension** (pulling) in the direction of its length. It is evident that the strength of a rod to resist tension depends upon the tenacity of its fibers, and hence must be proportional to the number of those fibers, or to the area of the cross-section.

For instance, the tension of the rod C is found to be proportional to the area of the cross-section; that is, if a rod with cross-section area B will hold 10,000 lb., a rod A twice as thick, measuring the diameter, will hold 40,000 lb., because the area of A is four times the area of B.

Very often it is necessary for others than practical engineers to be able to find out the tension of ordinary



round wrought-iron rods. For ordinary purposes the following table will be found convenient:

Wrought-iron rods, one-inch cross-section area, carry safely a weight of 10,000 *lb.*

When screws are used at the ends, the threads lessen the diameter of the rod and reduce the strength.

EXERCISES

1. A wrought-iron rod one-half inch square will hold safely how many pounds?

2. An electric-light chandelier, weighing 5000 lb., is held up by a solid circular rod. What is the diameter of the smallest wrought-iron rod that can safely be used? (Give the answer in the decimal of an inch.)

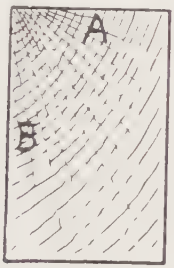
3. Give the largest safe loads for circular rods the diameters of which are as follows:

(*a*) Diameter of three-fourths of an inch.
(*b*) Diameter of one and one-eighth inches.
(*c*) Diameter of two inches.
(*d*) Diameter of two and one-half inches.

4. It is desired to suspend 21,000 lb. from a round rod of wrought iron. What shall be the diameter of the rod to carry the weight safely?

Lesson No. 35. Strength of Beams and Girders

To calculate the transverse strength of beams and girders supported at both ends and loaded in the center, the following rule will be found of large practical value:



A = breadth in inches

B = depth in inches

$$\frac{A \times B \times B}{\text{Span in feet}} \times \text{constant} = \text{breaking weight in tons}$$

The breadth of the beam in inches multiplied by the square of the depth in inches, divided by the clear span in feet, and the result multiplied by the following constants will give the breaking weight in net tons (2000 lb.):

TABLE OF CONSTANTS

Cast iron	1.10
Wrought iron	1.25
Ash	.325
White pine	.225
Yellow pine	.275
Oak	.30
Hemlock	.20

EXERCISES

1. The joists of the floor of a factory intended for machinery are of ash, 4 in. wide, 6 in. deep, and 9 ft. between the end supports; what is the greatest weight of

machinery that can be supported safely by 10 such joists? (Give the answer in tons.)

2. What is the greatest weight in pounds that a cross bar of wrought iron 2 in. wide, 3 in. deep, and 12 ft. between supports, will sustain safely?

3. What weight in pounds can be sustained safely hanging from a hemlock scantling 3 in. by 3 in., placed in a horizontal position, 10 ft. between the supports?

Lesson No. 36. Areas of Ellipses

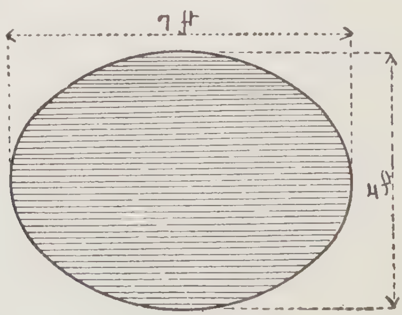
In ornamental work, as of floors and ceilings, it is often convenient to be able to find the area of the circular figure known as an **ellipse**.

To find the area of an ellipse use the following rule:

Multiply the longest diameter by the shortest; then multiply by 11 *and divide by* 14.

ILLUSTRATIVE EXERCISE

Find the area of an ellipse whose dimensions are in the following diagram:



The area will be found as follows:

$$\frac{7 \times 4 \times 11}{14} = 22 \text{ sq. ft.}$$

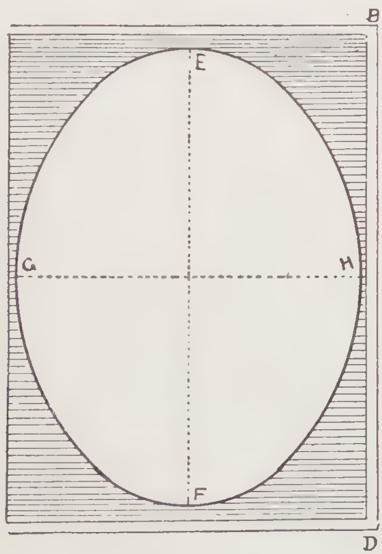
The rule can also be expressed thus:

Multiply the longest semi-diameter by the shortest semi-diameter, and the product by $\frac{22}{7}$.

EXERCISES

1. Find the area of an ellipse whose diameters are 21 ft. and 15 ft.

2. A race-course is laid out in the form of an ellipse three-quarters of a mile in longest diameter and 810 yd. in shortest diameter. Find the area in acres of the ellipse thus formed.



3. A tile floor is in general design similar to that shown in the diagram. From the character of the work it is necessary to estimate separately the elliptical center. EF is 63 ft. and GH is 45 ft. The entire floor is 75 ft. by 57 ft. The part of the floor outside of the ellipse costs \$4.90 a

square yard, while the elliptical section costs $ 3.20 a square yard. Find the entire cost of the floor.

Lesson No. 37. Simple Mechanics — Levers

A mechanical instrument or **machine** is a contrivance for making a force which is applied at one point available at some other point.

The simplest machines are rods used in pushing and ropes used in pulling; but what are called the *simple machines* or *simple mechanical powers* are the following:

1. The lever.
2. The wheel and axle.
3. The pulley.
4. The inclined plane.
5. The screw.
6. The wedge.

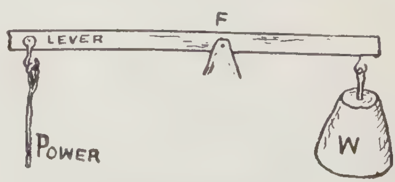
The force applied to a machine to set it in motion is called the **power**, and the resistance to be overcome is called the **weight**.

The efficiency or working power of a machine is measured by the fraction

$$\frac{W}{P} \text{ or } \frac{weight}{power},$$

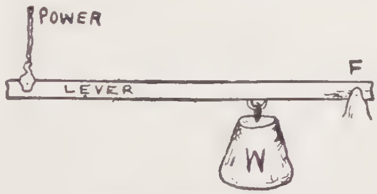
which is often called the **modulus** of the machine. When W is greater than P the machine is said to work at a mechanical advantage, and when W is less than P at a mechanical disadvantage.

A **lever** is a rigid rod, capable of turning round a fixed point in the rod; this point is called the **fulcrum.**



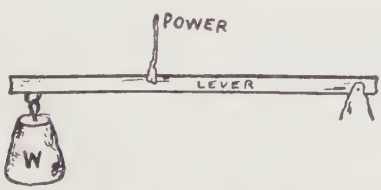
LEVER OF FIRST CLASS.

The illustration shows what is called a lever of the *first class.* The power and weight are on opposite sides of the fulcrum *F*, and act in the same direction. A poker between the bars of a grate, raising the coals, a spade, a pair of scissors, a common balance or scales, are examples of levers of this class.



LEVER OF SECOND CLASS.

Here we have an illustration of a lever of the *second class.* The power and weight are applied on the same side of the fulcrum, and act in opposite directions, the power being applied at a greater distance from the fulcrum than the weight is. A wheelbarrow is an illustration, the point where the wheel touches the ground being the fulcrum; an oar of a boat is another example, the blade of the oar in the water being the fulcrum.



LEVER OF THIRD CLASS.

In levers of the *third class* the power and weight are applied on the same side of the fulcrum, and act in opposite directions, the power being nearer to the fulcrum than the weight is.

General Rule: *The power or force multiplied by its distance from the fulcrum is equal to the weight multiplied by its distance from the fulcrum.*

ILLUSTRATIVE EXERCISES

1. In a lever of the first class the weight is 9 in. and the power is 12 in. from the fulcrum. If the weight is 4 lb., what is the power?

12 times the power $= 9$ *times the weight.*
12 times the power $= 9 \times 4 = 36.$
Therefore the power $= 36 \div 12 = 3$ lb.

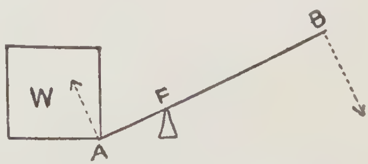
2. In a lever of the second class what power 15 ft. from the fulcrum will lift a weight of 600 lb. 3 ft. from the fulcrum?

15 times the power $= 3$ *times the weight.*
15 times the power $= 3 \times 600 = 1800.$
Therefore the power $= 1800 \div 15 = 120$ lb.

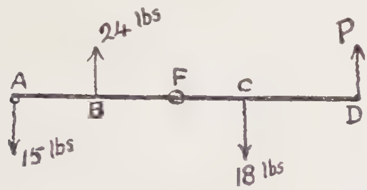
EXERCISES

1. If 480 lb. be applied to the end of a lever of the first class 135 in. from the fulcrum, what weight will it lift 45 in. from the fulcrum?

2. In a lever of the second class what power 14 ft. from the fulcrum will lift a weight of 441 lb. 4 ft. from the fulcrum?

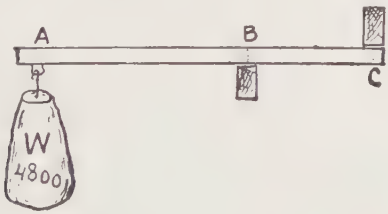


3. If AF is 3 ft. and AB is 13 ft., a pressure of 528 lb. (at right angles) at B will exert an upward pressure of how many pounds (at right angles) at A?



4. *AF* is 18 in.; *BF* is 6 in.; *FC* is 15 in.; *FD* is 36 in. What upward power at *D* will keep the lever in equilibrium?

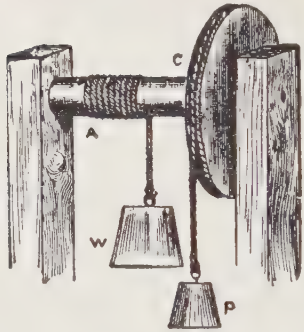
NOTE. — The difference between the products of the weights and arms at one side should equal the difference of the products of the weights and arms at the other.



5. A weight of 4800 lb. is suspended from a point *A* of a beam *AC*, 35 ft. long. *AC* rests on a cross beam at *B*, and the end *C* is under another cross beam. If *BC* is 14 ft., find the upward pressure in pounds at *C*. (Weight of *AC* and breadth of cross beams not considered.)

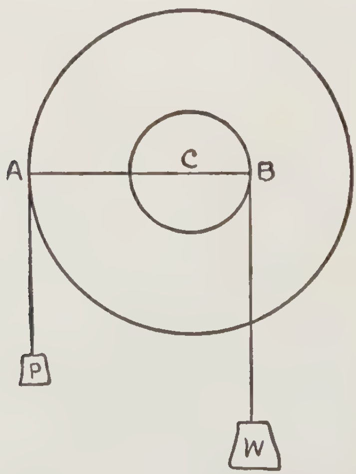
Lesson No. 38. Simple Mechanics — The Wheel and Axle

The **wheel and axle** may be described as a continuous lever. It consists of a wheel, to the outer edge of which the power is applied, and of the axle, to which the wheel is fastened and to the circumference of which the weight is attached by means of a cord.



WHEEL AND AXLE.

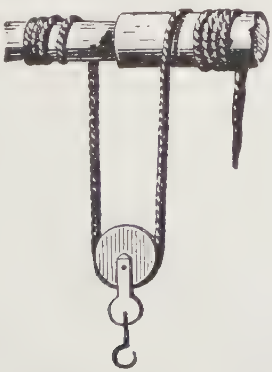
The above illustration shows the machine in its most elementary form. P and W represent the power and the weight. In practice, as you have frequently seen, the wheel disappears, and you simply have the crank left; as, however, this is turned around, it describes the same path as a continuous wheel would do; therefore its principle is the same.



Note the drawing of a cross-section of the *wheel and axle*. AB is the lever with the fulcrum at C. From the fact that when the power and the weight act as shown in the diagram, there is always a lever in the position AB, the machine is called a "continuous lever." The radius of the wheel forms the long arm of the lever, and the radius of the axle the short arm. If the radius of the wheel is five times the radius of the axle, the relation of P to W will be as 1 to 5; that is, a weight of 1 lb. at the circumference of the wheel will balance a weight of 5 lb. at the circumference of the axle.

You can also readily see that for one complete turn of the wheel we have a complete turn of the axle; so that a weight running down a distance equal to the circumference of the wheel will lift a weight through a distance equal to the circumference of the axle.

As the circumference of wheel and axle bear the same proportions as their radii, it is clear that for every inch through which the weight is raised the power will have to run down 5 in., and the rate at which the weight is raised is only one-fifth that at which the power runs down. Here we have the *law of virtual velocities* — what we gain in power we lose in speed.

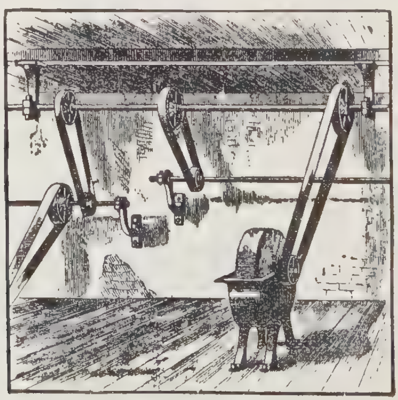


DIFFERENTIAL WHEEL AND AXLE.

If the axle is made of two different thicknesses, as shown in the diagram, it is called the **compound** or **differential wheel and axle.** From the figure you will notice that the

CAPSTAN.

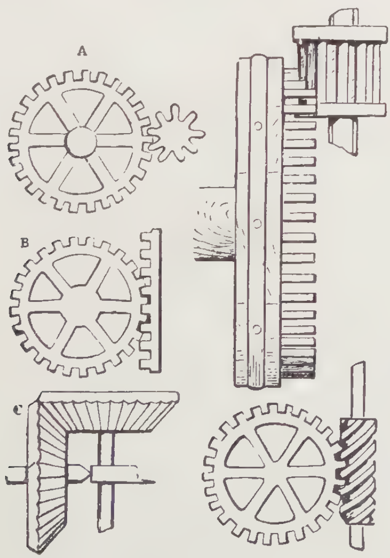
cord coils in one direction on one section, and in the opposite direction on the other, so that when it is winding on to the larger part it is unwinding from the smaller. The load raised will, for every complete turn of the axle, be lifted



ENDLESS BELTS.

through a height equal to half the difference of the circumferences of the larger and smaller axles. The power developed in a machine of this character will be more fully illustrated in the lesson on the pulley.

The wheel and axle in its elementary form is not so frequently met with as modifications of it, for the greater part of wheel-work machinery comes under this principle. We still, however, find it employed in raising water and loads from the holds of ships in the form of a windlass. A much more powerful application of the wheel and axle is that of the capstan, where the block is vertical. By this means enormous anchors and chain cables are raised with ease. The steering wheel which works the ship's rudder is another application of the same principle. Whether in compound or simple machinery, no mechanical power, perhaps, admits of so many modifications and uses as the wheel and axle.



TOOTHED WHEELS.

For example, this power occurs in toothed wheels or cogged wheels, and in wheels connected by endless belts, in cranes, watches, steam engines, and machinery of all kinds. In all these applications of the principle, the relation of the power to the weight is the same as that given for the ordinary *wheel and axle*. You can easily understand that in ordinary practice, however, the friction between wheels armed with teeth is very great, so that much of the power applied is really lost in overcoming this.

GENERAL RULE: *Multiply the power at the edge of the wheel by its radius, and divide the product by the radius of the axle. The quotient is the weight that the power will raise.*

This rule is practically the same as that stated for the application of the lever.

EXERCISES

1. Find the power necessary to raise a weight of 400 lb. by an axle of 10 in. and a wheel of 50 in. in diameter.

2. If the radius of the wheel is 3 ft., the weight 18 lb., and the power 3 lb., what is the radius of the axle?

3. If the radius of the wheel is 6 ft., the radius of the axle 2 ft., and the weight 36 lb., what must be the power to produce equilibrium?

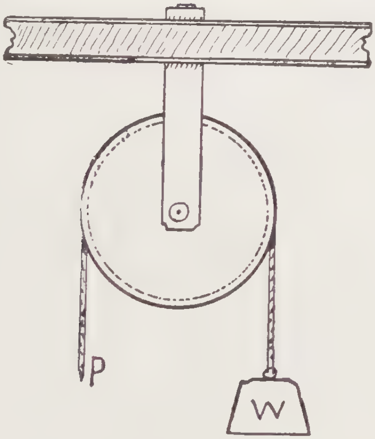
4. If in a capstan the radius of the axle is 1 ft., and 6 men push, each with a force of 100 lb., on spokes 5 ft. long, how many pounds will they be able to support?

5. The radius of the wheel being three times that of the axle, and the cord on the wheel being only strong enough to support a tension equivalent to 36 lb., find the greatest weight which can be lifted.

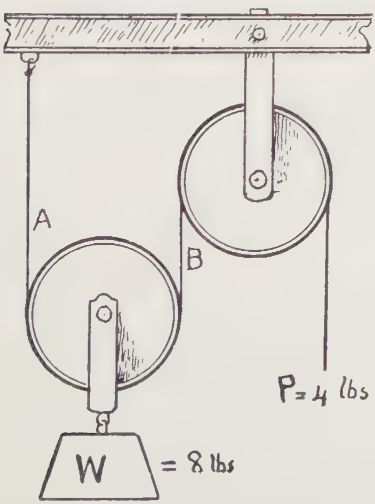
Lesson No. 39. Simple Mechanics — Pulleys

The **pulley** is a wheel over which a cord, or chain, or cable, is passed, in order to transmit the force applied to the cord in another direction.

When the block in which the pulley turns is fixed, the pulley is said to be **fixed**. There is no mechanical advan-



FIXED PULLEY.



MOVABLE PULLEY AND FIXED PULLEY.

tage gained by the use of one or more fixed pulleys; but this contrivance is of the greatest use in enabling us to *change the direction of the force*. Thus, it is much more convenient to raise a bucket from a well by drawing downward, as is the case where the rope passes over a fixed pulley above the head, than by drawing upward, leaning over the curbing.

A PULLEY WITH DOUBLE BLOCKS.

From its portable form, its cheapness, and the facility with which it can be applied, the pulley is one of the most convenient and most useful of the mechanical powers.

When the pulley block moves up and down with the cord, it is called a **movable** pulley; and when movable and fixed pulleys are worked together, we get what is called a "system of pulleys."

In the foregoing illustration suppose that the weight is 8 lb. It is supported by two cords, A and B; that is, the two sections of the cord support 4 lb. each. Now the cord being continuous, the power must be 4 lb.

We leave out of consideration the weight of the pulley in this case, and also the friction of the various parts of the machine.

Notice how the *law of virtual velocities* comes in here. We have seen that the weight is sustained by two cords; if, therefore, it has been raised one foot, each cord must be shortened one foot. To do this, the power must run down two feet. To get the full value of this machine the cords must be parallel.

Notice that in this system of pulleys we may have more than one movable pulley. In the foregoing illustration, suppose that where the weight W is now attached, a cord were fastened which passed under a second movable pulley and up to the beam further along than A. Then this cord would have a tension of 8 lb., and therefore would support a weight attached to its pulley of 16 lb. We thus see that in this case the selection of P to W would be as 1 to 4, and the distance through which P would have to travel would be 4 times that through which W would be raised.

If we increased the number of movable pulleys to three, the relation of P to W would be as 1 to 8, and the distance through which P would have to travel would be eight times that through which W would be raised.

In this system of pulleys the weights of the blocks of the movable pulleys act against the power.

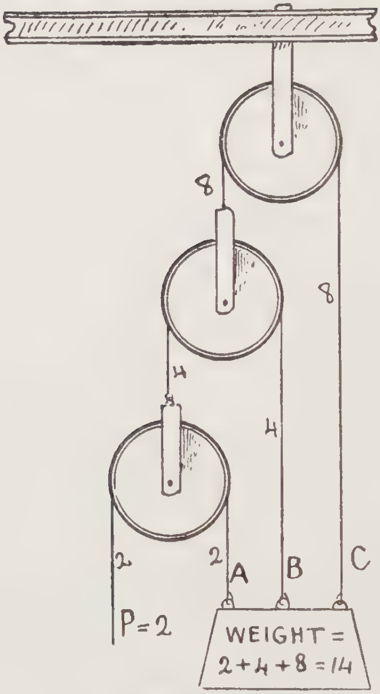
In a system as shown in the above illustration in which the upper block contains two pulleys and the lower block two pulleys, the relation of P to W is as 1 to 4. The weight being sustained by four cords, each bears a quarter, and a weight of 4 lb. will be kept in equilibrium by a power of 1 lb.

The blocks used in a system of this character are called **single** if there is one pulley in each, **double** if there are two pulleys in each, **treble** if there are three pulleys in each, and **quadruple** if there are four pulleys in each.

In the system of pulleys shown in the above illustration each rope is attached to the weight. Suppose that the power is 2 lb.; then rope A will hold 2 lb.; the power and A together will make a tension of 4 lb. on rope B; this doubled will give 8 lb., or the tension on rope C; so that A, B, and C together will hold a weight of $2 + 4 + 8$, or 14 lb. The weights of the pulleys are not considered.

In this system of pulleys the weights of the blocks of the movable pulleys add to the power.

Pulleys are used very generally, especially in building

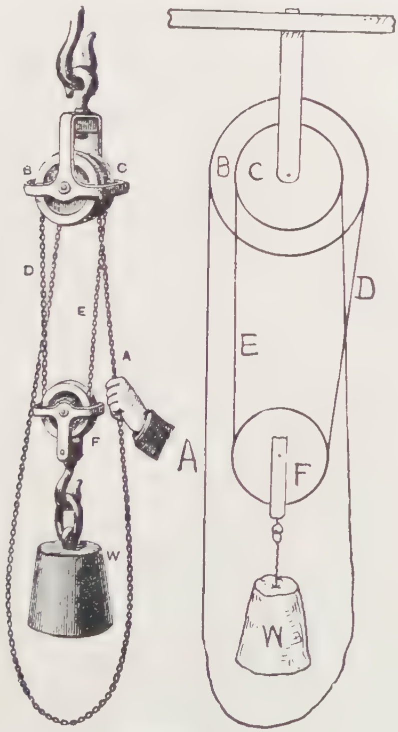


A SYSTEM OF PULLEYS.

operations, where heavy beams, girders, or blocks of stone have to be raised. On board ship it is the favorite mechanical power by which rigging is raised, and cords and ropes are tightened, and goods lifted from or lowered into the hold.

One of the most ingenious applications of this mechanical power is called the **differential pulley**. It is a compact and powerful machine. When the chain *A* is pulled, it turns the two pulleys, or rather one pulley with two grooves, *B* and *C*. Now *C* is a little smaller than *B*, so that although at *D* the chain is lifted, it is lowered at *E*. If the circumference of *B* is 20 in., and that of *C* is 19 in., then when *A* is pulled

20 in., D is lifted 20 in., but E is lowered only 19 in.; hence, the pulley F, although it will turn a considerable distance, will only rise one-half of an inch, carrying the



DIFFERENTIAL PULLEY.

weight W with it. From this experiment — applying the principle of vertical velocities — we can deduce the following rule for a differential pulley block:

If $A =$ *circumference of larger groove,*

and $B =$ *circumference of smaller groove,*

then $\quad Power \times A = Weight \times \frac{1}{2}(A - B).$

Or, if it is not convenient to get the circumferences, the result will be the same if the diameters are used. The friction present in a differential pulley largely reduces the lifting value of the power.

ILLUSTRATIVE EXERCISES

1. In a system of two pulleys, as shown in the second illustration, what power will support a weight of 60 lb.?

$$P = \frac{W}{2} = \frac{60}{2} = 30 \text{ lb.}$$

2. With pulley blocks in a system similar to that shown in the third illustration, but with three pulleys in each block instead of two, what power will support a weight of 3000 lb.?

$$P = \frac{W}{\text{No. of Ropes}} = \frac{W}{6} = \frac{3000}{6} = 500 \text{ lb.}$$

3. In a system of pulleys similar to that shown in the fourth illustration, but with five pulleys, what power will support a weight of 930 lb.?

$$P \textit{ is to } W \textit{ as } 1 \textit{ is to } 31.$$

$$P = \frac{W}{31} = \frac{930}{31} = 30 \text{ lb.}$$

4. A differential pulley block contains a pulley the larger groove of which is $9\frac{3}{4}$ in. in diameter, and the smaller groove of which is $9\frac{1}{2}$ in. in diameter. What power will support a weight of 6240 lb.?

$$\textit{Power} \times 9\tfrac{3}{4} = \textit{Weight} \times \tfrac{1}{2}(9\tfrac{3}{4} - 9\tfrac{1}{2}).$$

$$\textit{Power} \times 9\tfrac{3}{4} = \textit{Weight} \times \tfrac{1}{8}.$$

$$\textit{Power} \times 9\tfrac{3}{4} = 6240 \times \tfrac{1}{8} = 780 \text{ lb.}$$

$$\textit{Power} = 780 \text{ lb.} \div 9\tfrac{3}{4} = 80 \text{ lb.}$$

In these exercises and in the exercises which follow, except where otherwise stated, the weight of the pulleys and the friction of the cables are not considered.

Lesson No. 40. Simple Mechanics — Pulleys (Continued)

EXERCISES

The following exercises refer to the style of pulley system shown in the second illustration of Lesson No. 39:

1. If there are four movable pulleys, what power will support a weight of 480 lb.?

2. In a system of six movable pulleys, what power will support a weight of 1280 lb.?

3. If there are four movable pulleys, whose weights commencing with the highest are 1 lb., 2 lb., 4 lb., and 8 lb. respectively, and the weight is 160 lb., find the power.

4. If there are five movable pulleys, whose weights commencing with the highest are 1 lb., 2 lb., 4 lb., 8 lb., and 16 lb., what power will support the pulleys and an additional weight of $642\frac{1}{2}$ lb.?

Lesson No. 41. Simple Mechanics — Pulleys (Continued)

EXERCISES

The following exercises refer to the style of pulley systems shown in the third and fourth illustrations of Lesson No. 39:

1. In a system of triple-block pulleys, what power will support a weight of 3600 lb.?

2. In a system of double-block pulleys, what power will support a weight of 4800 lb.?

3. In a system of triple-block pulleys a power weighing 160 lb. rests on the floor; the weight is 72 lb. Find the pressure in pounds of the power on the floor.

4. In a system of four pulleys as shown in the fourth illustration (Lesson No. 39), what power will support a weight of 300 lb.?

Lesson No. 42. Simple Mechanics — Pulleys (Continued)

EXERCISES

1. A differential pulley block contains a pulley the larger groove of which is $8\frac{1}{2}$ in. in diameter and the smaller groove of which is $8\frac{3}{8}$ in. in diameter. What power will support a weight of 2720 lb.?

2. A differential pulley block contains a pulley the larger groove of which is $16\frac{3}{4}$ in. and the smaller groove of which is 15 in. in diameter. What power will support a weight of 1072 lb.?

3. A differential pulley block contains a pulley the larger groove of which is 20 in. in circumference and the smaller groove of which is 19.99 in. in circumference. The power is 100 lb.; what weight will it support?

4. A differential pulley block used in hoisting safes contains a pulley the larger groove of which is $6\frac{7}{8}$ in. in diameter and the smaller groove of which is $6\frac{3}{4}$ in. in diameter. What power will support a safe weighing 2200 lb.?

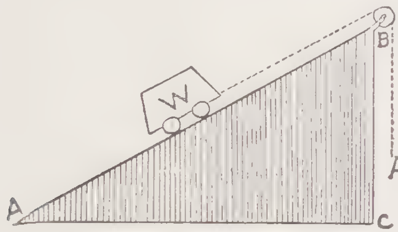
NOTE. — When the weight is at rest, the power needed is not nearly so large, owing to the great holding power of this system, due to friction. In the foregoing exercises the power of friction is not to be considered.

Lesson No. 43. Simple Mechanics — Inclined Planes

The *inclined plane* is put down as fourth of the simple machines. It is perhaps one of the oldest of the mechanical powers, and it is frequently applied without any idea of its belonging to the class called *machines.* In early times building operations were most likely carried on by arranging inclined paths, up which heavy material was raised to any height required. This inclined surface was probably used before any such contrivance as frame or pole scaffolding was thought of.

All our cutting tools come under the same class, and are

applications of this simple mechanical power — the knife in cutting a pencil, as well as the wedge with which we split a block. In the screw we simply wind an inclined plane upon itself, and we get this secondary mechanical power, which, combined with the lever, forms one of the most powerful tools we have in use.



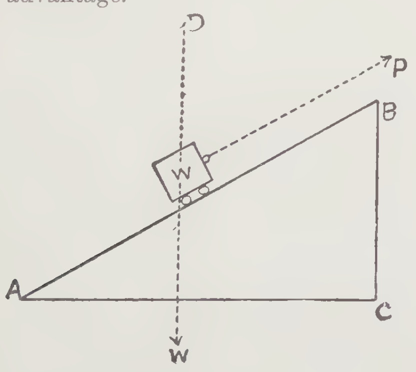
AN INCLINED PLANE.

In this illustration if *BC* is 6 in. and *AB* is 24 in. in length, the relation of the power to the weight is as the height of the plane is to the length of the incline.

$$P \times Length = W \times Height.$$

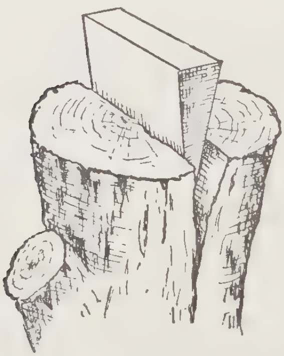
Therefore $\dfrac{Length\ of\ Plane}{Height\ of\ Plane} = \dfrac{W}{P} =$ mechanical advantage.

From this it will be seen that the greater the length of the plane compared with the height, the greater the mechanical advantage.



The weight acts in the direction of gravity or in a line perpendicular to the base, while the force, or power, acts parallel to AB. If the power acted in the direction of D, then the power and weight would necessarily be equal to counteract each other.

One of the most useful applications of the inclined plane is the wedge. It is an inclined plane doubled, and a large resistance is overcome by a very small force. The longer the wedge is, compared with its breadth, the easier the work is done. The mechanical advantage of the wedge is obtained by dividing its length along the middle by its thickness at the broad end. If, for illustration, the length of a wedge is 10 in. and the thickness 2 in., the mechanical advantage is 5. If such a wedge were struck with a blow of 40 lb., it would exert a splitting power of 5×40, or 200 lb.



A WEDGE.

The ax, the hatchet, and the adz carry with them the principle of the wedge as well as that of the hammer. The carpenter's chisel is another familiar instance.

Lesson No. 44. Simple Mechanics — Inclined Planes (Continued)

EXERCISES

NOTE. — In these exercises the resistance of friction is not considered.

1. Find the power necessary to raise 1280 lb. up an inclined plane 8 ft. long and 5 ft. high.

2. The length of an inclined plane is 15 ft.; the perpen-

dicular height is 6 ft. What force will be required to sustain a weight of 150 lb.?

3. The length of an inclined plane is 20 ft.; the perpendicular height is 6 in. What power will be required to keep a loaded car weighing 4800 lb. from rolling down the plane?

4. A train weighing 160 tons is moving up a grade of 5 ft. to the mile. What power in addition to that necessary on a level will be needed?

5. A wedge is $3\frac{1}{2}$ in. thick and 14 in. long. What splitting force will be exerted by a blow of 36 lb.?

6. A railway train weighing 50 tons is drawn up an incline of 1 ft. in 30 ft. by a cable. What is the least strain on the rope, if 10 lb. per ton be allowed for resistance to motion on level rails?

7. Which will support the greater weight, a power acting horizontally or the same power acting parallel to the plane?

Lesson No. 45. Simple Mechanics — The Screw

The **screw** is an inclined plane wound on a cylinder. If you will take a piece of paper of the shape of an inclined plane and wrap it around your lead pencil, beginning with the wide end of the plane, you will readily see the relation between the screw and the inclined plane.

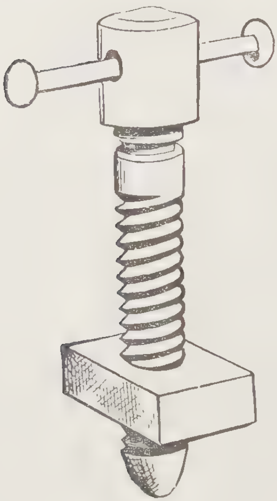
The longer the inclined plane, the nearer will be the threads, and consequently the greater will be the mechanical advantage.

The screw is of no use as a mechanical power unless it has a fixed block or is itself fixed. The block in which the screw works is called the **nut**.

The screw is rarely used to raise weights, but we are all very familiar with its use in binding things together, and we know how tightly it holds them.

The mechanical advantage of the screw may be seen in examining what it does when it is used to raise weights.

When so used the power is usually applied at the end of a lever (or arm) and works constantly in the circumference of a circle of which the lever (or arm) is the radius, while the weight rests upon the head of the screw and acts vertically downward in the direction of the axis of the screw. It is evident that when the power acts continuously through a complete revolution of the lever, the weight is raised upwards a distance equal to the distance between two threads of the screw. Applying the principle of vertical velocities, we have the following relation: *The power multiplied by the circumference of the circle traversed by the end of the lever is equal to the weight multiplied by the distance between two threads of the screw ;* or



A SCREW.

$$P \times C = W \times d$$

where C is the circumference of the circle referred to, and d the distance between two threads.

ILLUSTRATIVE EXERCISE

Let us take an example. Suppose the distance between two threads be half an inch, and the length of the lever $24\frac{1}{2}$ in., and the power 100 lb. We have, to find the pressure, or W:

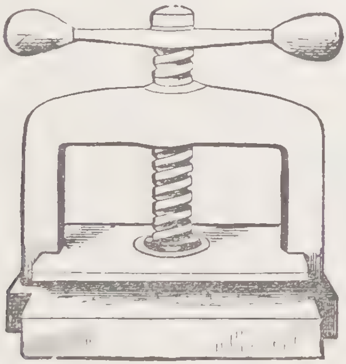
$$Lever \times 2 = Diameter = 49 \text{ in.}$$

Therefore $$W \times \tfrac{1}{2} = (49 \times \tfrac{22}{7}) \times 100$$

or, $$W = (49 \times \tfrac{22}{7}) \times 100 \div \tfrac{1}{2} = 30{,}800 \text{ lb.}$$

That is to say, a power of 100 lb. at the end of such a lever will exert a pressure of over 15 tons.

One of the most common applications of the principle of the screw is seen in the screw-press used by bookbinders, printers, etc., and in the common copying press used generally for copying letters.



A SCREW-PRESS.

As an example of a weight-lifter, we have none of more importance than the ordinary screw-jack, which sometimes plays an important part in lifting a house, in launching a tardy ship, or in hoisting a locomotive back on the track. It acts much in the same way as the screw-press we have just described. The common screw-vise is another example of the same mechanical power.

EXERCISES

1. A screw-press has a double lever, each arm being 14 in. in length; the distance between the threads is three-quarters of an inch. What pressure will be exerted at the head of the screw by a power of 50 lb. applied at the end of each lever?

2. A screw-jack in the shape of a capstan has 5 levers, each 7 ft. long; the distance between the threads of the screw is 1 in. What lifting pressure will be exerted if 1 man shoves on each lever with a power of 120 lb.?

Lesson No. 46. Simple Mechanics — Review

EXERCISES

1. If 960 lb. be applied at the end of a lever of the first class 75 in. from the fulcrum, what weight will it lift 12 in. from the fulcrum?

2. Find the power necessary to raise a weight of 800 lb. by an axle 7 in. and a wheel 35 in. in diameter.

3. If in a capstan the radius of the axle is 7 in. and eight men push, each with a force of 120 lb., on eight spokes, each 6 ft. long, how many pounds will they be able to support?

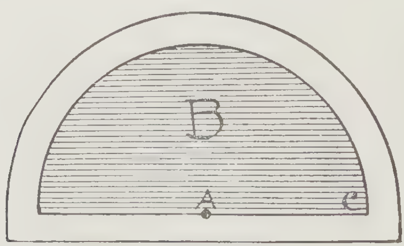
We have now completed our lessons in industrial arithmetic, including simple mechanics, and before giving some explicit instructions as to the forms of bids, estimates, and specifications we shall take a general review of the work already covered.

Lesson No. 47. General Review

EXERCISES

1. A piece of land is 3 mi. long by 180 rd. wide. How many acres does it contain?

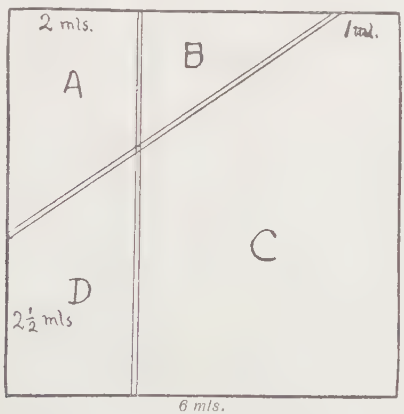
2. A street $1\frac{1}{4}$ mi. long is to be paved with asphalt. The asphalt is to reach from curb to curb, a distance of $1\frac{1}{2}$ ch. Find the expense of paving at $ 2.20 a square yard.



3. A grass plot *B* is in the form of a semicircle, *A* being the center. The radius is 126 ft.; that is, from *A* to *C*. The plot is surrounded by a walk 12 ft. wide. Find the cost of paving the walk at $ 1.80 a square yard.

4. An athletic field is in the form of a circle in area $15\frac{2}{5}$ acres. It is surrounded by a tight board fence 12 ft. high. Find the cost:

(1) Of the inch lumber in the fence at $ 20 per thousand.
(2) Of painting the fence both sides at 3¢ a square yard.



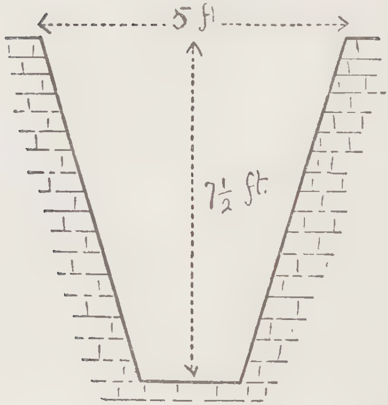
5. Two railways cross a Dakota township as shown in the map. Find the area of each of the four divisions marked *A*, *B*, *C*, and *D*. The township is a square.

6. What weight of water will a rectangular tank contain, the length being $6\frac{1}{2}$ ft., the breadth 25 in., and the depth 2 ft.? (A cubic foot of water weighs $62\frac{1}{2}$ lb.)

Lesson No. 48. General Review (Continued)

EXERCISES

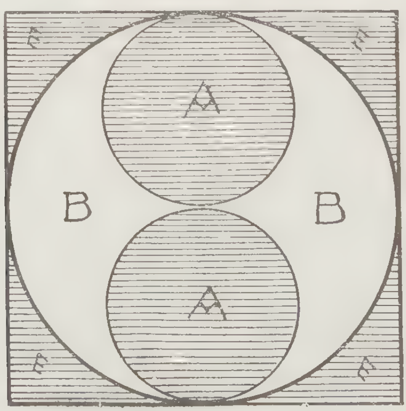
1. A pile of lumber is 14 ft. high, 24 ft. long, and 15 ft. wide. Considering that one-third of the space occupied is air, what is the lumber worth at $ 30 a thousand?



2. The cross-section of a ditch is as shown in the diagram; namely, 5 ft. wide at the top, 2 ft. wide at the bottom, and $7\frac{1}{2}$ ft. deep. Find the cost of the excavation per rod at 30¢ a load.

3. How many feet of lumber in a chute 2 ft. square (outside measurement), made of 3-in. plank, and 48 ft. long? (Exact measurement.)

4. A pile of tanbark is 36 yd. long, 18 ft. high, and 20 ft. wide. What is it worth at $ 6.40 a cord?



5. An inlaid square floor is 42 ft. on each side and is laid out in circles as shown in the diagram. The sections marked E are to cost 40¢ a square foot; the sections marked B are to cost 50¢ a square foot, and the sections marked A are to cost 60¢ a square foot. Find the entire cost of the floor.

6. Find the cost of plastering the four walls and ceiling of a hall 60 ft. by 48 ft. and 21 ft. high, at 20¢ a square yard, allowing nothing for openings.

Lesson No. 49. General Review (Continued)

EXERCISES

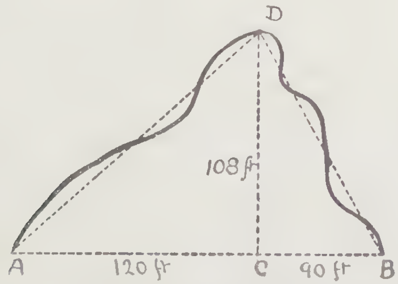
1. Find the number of perches in a stone wall 108 yd. long, 11 ft. high, and 2 ft. thick.

2. How many perches of stone in a rectangular foundation 36 ft. by 21 ft., outside measurement, the wall being 10 ft. high and $1\frac{1}{2}$ ft. thick?

3. If railway ties are 3 ft. apart from center to center, how many ties are necessary for a road $\frac{3}{4}$ of a mile long?

4. A city council decides to spend $60,000 in asphalt sidewalks. They find that the cost will be $2 a square yard. If the walks are to be 4 ft. wide, how many miles of sidewalks can they build? (Give the exact answer.)

5. How many feet of lumber in a circular platform 308 ft. in circumference and 2 in. thick?



6. A railway contractor finds it necessary to cut through a hill, the cross-section dimensions of which are as shown

in the diagram. The cutting must be 4 rd. long. If gravel cars carry eight loads of earth, how many carloads must be removed?

Lesson No. 50. General Review (Continued)

EXERCISES

1. The map of an eastern township is drawn on a scale of 4 in. to the mile. The map is $4\frac{1}{2}$ ft. by $3\frac{1}{4}$ ft. How many acres are there in the township?

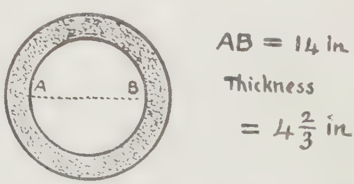
2. The ground plan of a rectangular building is drawn upon a scale of 4 ft. to a half inch. The drawing measures $12\frac{1}{2}$ in. by $8\frac{1}{2}$ in. What are the length and breadth of the building?

3. If a bushel of grain occupies $1\frac{1}{4}$ cu. ft., how deep must the grain be in a 15-ft. by 6-ft. bin to contain 400 bu.?

4. Find the cost of cementing a circular cistern, including the bottom, which measures $5\frac{1}{2}$ ft. in diameter and 12 ft. deep, at 75¢ a square yard.

5. What will it cost to lay 60,000 bricks, including cost of bricks at $7.20 per thousand, 4 bbl. of lime at $1.40 a barrel, 42 bu. of sand at 8¢ a bushel, and labor at $3.20 a day of a man who can lay 1200 bricks a day, with helper at $1.50 a day?

6. An eight-roomed city house, three stories, has an 18 ft. frontage and is 48 ft. deep, and costs $2073.60. What is the rate per square foot of floor?



7. It is required to find the weight in tons of the metal pipe of the dimensions shown, necessary to stretch one mile, having given that a cubic foot weighs 800 lb.

Lesson No. 51. General Review (Continued)

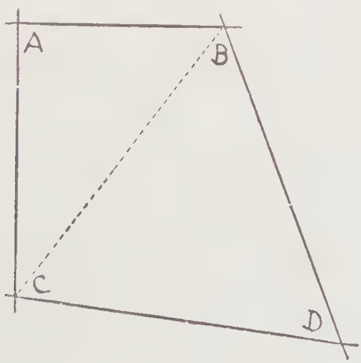
EXERCISES

1. An eight-roomed city house, three stories, 18 ft. frontage and 48 ft. deep, costs $1296 to build. What is the rate per square foot of floor?

2. Find the area of a triangle the sides of which are 300 ft., 240 ft., and 290 ft.

3. On an estimate basis of 70¢ per square foot and 15% added for extras, find the cost of building a row of 12 houses, each of three stories, and each containing 720 ft. to a floor.

4. Find the value at $18 per thousand of the 2-in. lumber necessary in building a sidewalk 9 ft. wide around a square, the inside measurement of which is 12 rd. on a side.



5. Four streets of a city form a lot of the shape *ABCD*, shown in the diagram. The angle at *A* is a right angle. *BC*, *CD*, and *BD* are the same length. *AC* is 20 rd. and *AB* is 15 rd. Find the cost of filling in the entire lot to a depth on an average of 6 ft., at $3\frac{1}{2}$¢ a load.

6. Find the cost of painting 240 doors, both sides, each 4 ft. 3 in. by 9 ft., at 21¢ a square yard.

7. If wall paper is 18 in. wide and a single roll is 24 ft.

long, how many rolls will be required for the four walls of a hall 36 ft. by 18 ft. and 12 ft. high, deducting five rolls for windows?

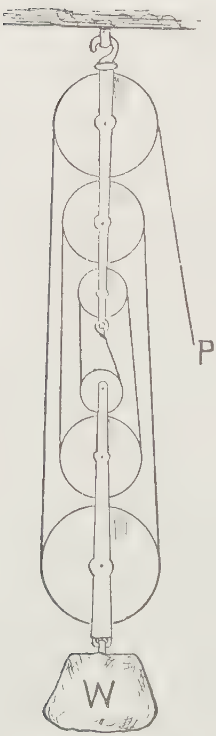
8. Find the number of feet, board measure, in 25 joists, each 6 in. by 3 in. and 18 ft. long.

9. How many square yards of block pavement in a street half a mile long and $1\frac{1}{2}$ ch. wide?

10. I want to cut an acre of land from a long, narrow field which is 55 yd. wide. What length of it must I take?

Lesson No. 52. General Review (Continued)

EXERCISES



1. If in a capstan the radius of the axle is 12 in., and eight men push, each with a force of 120 lb., on spokes 6 ft. long, how many pounds will they be able to support?

2. In a system of pulleys similar to that shown in the figure, if the weight is 47 lb. and the weight of the lower pulley-block is 3 lb., find the power.

3. In a system of pulleys similar to the foregoing, if there are five pulleys in each block, what weight can be lifted by a power of $137\frac{1}{2}$ lb.?

4. An iron bar 24 ft. long rests on a pivot 6 ft. from one end. What power at the end of the longer arm will be necessary to hold two weights, one of 1200 lb. at the end of the shorter arm and the other of 2500 lb. hanging on the shorter arm $2\frac{1}{2}$ ft. from the pivot?

5. A differential pulley-block contains a pulley the larger groove of which is 8 in. in diameter and the

smaller groove $7\frac{3}{4}$ in. in diameter. What power will support a weight of 6400 lb.?

6. Find the power necessary to raise 1449 lb. on an inclined plane 9 ft. long and 3 in. high.

7. Find the power necessary to raise a weight of 2100 lb. by an axle 6 in. and wheel 42 in. in diameter.

Lesson No. 53. General Review (Continued)

EXERCISES

1. A map 6 ft. long and 4 ft. wide represents 13,824 sq. mi. of country. To what scale is it drawn?

2. There are 107 sq. yd. in the surface of the walls and ceiling of a room 15 ft. by 18 ft. How high is the room?

3. A farmer has a bin 18 ft. long, 10 ft. wide, 6 ft. deep, which is $\frac{3}{4}$ full of wheat. If a bushel equals $2150\frac{1}{2}$ cu. in., what is the wheat worth at 90¢ a bushel?

4. How many square miles of country will be represented by a map $10\frac{1}{2}$ in. long and 6 in. wide drawn on a scale of 3 mi. to an inch?

5. A garden, 180 ft. long by 150 ft. wide, is surrounded by a tight board fence 6 ft. high. What will it cost to paint the fence, both sides, at 12¢ per square yard?

6. How wide must a lot be to contain $\frac{1}{8}$ of an acre if it is 88 ft. long?

7. The length of a rectangular field whose area is $3\frac{3}{5}$ acres is to its width as 16 is to 9. Find the cost of fencing it at $ 3 a rod.

8. A contractor undertakes to dig a ditch $3\frac{1}{2}$ mi. long, 12 ft. wide at the top, 6 ft. wide at the bottom, and 5 ft. deep, for 25¢ per cubic yard. How much money should he receive for the work?

9. Find the cost of a pile of lumber 16 ft. long, 3 ft. 3 in. wide, and 5 ft. 6 in. high, at $ 22.50 per thousand.

10. Find the value of the lumber at $ 18 per thousand that will be required to build 500 yd. of plank sidewalk

10 ft. wide, 2 in. thick, and resting on three continuous lines of scantling 4 in. square.

11. The inside dimensions of a rectangular fort are 210 ft. and 180 ft. The stone wall surrounding this space is 5 ft. thick and 12 ft. high. How many cubic feet of masonry are there in the wall? (Exact measurement required.)

12. At 36¢ a square yard how much will it cost to plaster the walls and ceiling of a room 18 ft. long, 16 ft. wide, and $9\frac{1}{2}$ ft. high, allowing 73 sq. ft. for windows and door?

13. Find the length in inches of the edge of the largest square marble slab that can be used in flooring a room 33 ft. 10 in. long and 24 ft. 6 in. wide.

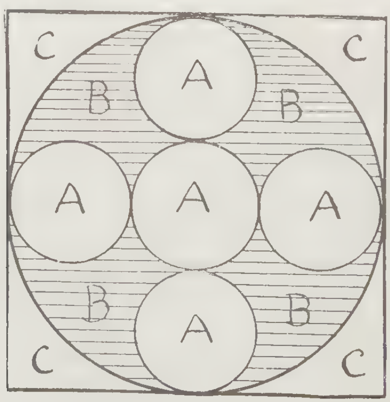
14. At 26¢ a cubic yard what will it cost to dig a cellar 36 ft. long, 28 ft. wide, and 7 ft. deep?

15. How many cords of wood can be piled in a shed 72 ft. long, 48 ft. wide, and 16 ft. high?

16. How many posts placed 8 ft. apart will be required to go around a quarter-section of land?

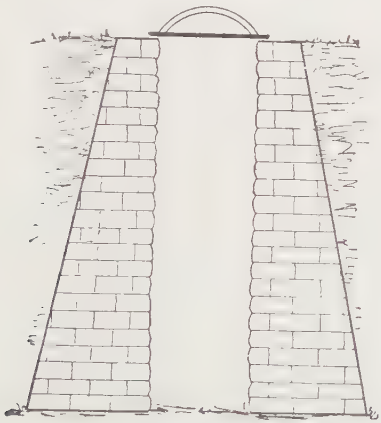
17. A reservoir is 40 ft. 6 in. long and 21 ft. 4 in. wide. How many cubic feet of water must be drawn off to lower the surface 5 in.?

18. The floor, walls, and ceiling of a room 18 ft. long, 16 ft. wide, and $12\frac{1}{2}$ ft. high, are made of inlaid walnut, oak, and



rosewood. Find the cost of the work at $2.50 a square foot for labor and 7½¢ a foot for lumber.

19. The court of a public building is a perfect square 42 ft. on each side. Its tile floor is laid out in circles, as shown in the drawing. The circles marked *A* are of the same size and cost $2 a foot; the sections marked *B* cost $1.40 a foot; the sections marked *C* cost $1.20 a foot. Find the entire cost of the floor.



20. Two stone abutments are to be built for a bridge. Their sides and fronts are perpendicular. The back of each slants as shown in the figure. They are 15 ft. wide, 24 ft. high, 9 ft. thick at the bottom and 3 ft. thick at the top. It is required to find the number of cubic yards of masonry.

This lesson completes the practical problems of this course.

Lesson No. 54. The Law of Contracts

NOTE. — In all important legal matters it is advised that competent legal authority be consulted. The notes and principles set down here will enable those who become familiar with them to avoid legal com-

plications arising from the careless drawing of agreements and specifications.

A contract is an agreement, duly entered into, by competent parties, for a legally sufficient consideration, to do or not to do some specific thing.

In order to constitute a binding contract there must be, therefore, an agreement (1) properly made by parties who are lawfully able to contract, (2) founded on a consideration legally sufficient, (3) for some stated object.

There are in general two kinds of contracts; namely, (1) sealed contracts, and (2) simple written or oral agreements, unaccompanied with the formality of a seal.

A sealed contract is a written agreement signed by the parties, the signatures having appended to them what is commonly known as a seal. In many states a mere scroll inclosing the word "seal" made opposite the name of the signer is sufficient, while in other states the seal has no longer any significance.

The principal difference between a sealed contract and one not under seal is that in the former case a valuable consideration is not required to support the agreement, while in the latter case the contract may be considered invalid unless such a consideration can be shown to exist. The affixing of a seal to a signature implies a special care and deliberation on the part of the signer, more than can be assumed in the case of a simple signature.

Each party to a contract must clearly express his assent before he is legally bound. Thus, a proposal by one party must be accepted by the other within the time allowed in the proposal, or a reasonable time, if none is stated; and the proposer must know of its acceptance.

The acceptance must be of the exact offer made, and if the offer is once rejected it can only be accepted afterwards on a renewal of the offer by the proposer.

In general, any person of legal age and sound mind is capable of making a binding contract.

The national or any state government may become a party to a contract, and such government may sue on its contracts and enforce them, but the converse of this is not true. Neither the United States nor any state can be sued without its consent. The only remedy in such a case is through an appeal to Congress or to the state legislature.

All public corporate governments, subordinate to that of the state, as of the county, or township, or city, can be sued upon their contracts.

Lesson No. 55. Offers and Acceptances

To make a contract the parties thereto must agree to the same thing in the same sense. If this is not done, the contract is not complete. Usually the assent must be given at the time of making the offer. If the person who receives a proposition or offer accepts it on any condition or with any change of the terms, even though this be not material, the contract is not complete unless the party who made the offer assents to the modification.

The agreement is not completed until each party has communicated to the other his intention in the matter. An acceptance by mail can be withdrawn by telegraph, provided the telegram is received before the letter.

A mere offer may be withdrawn at any time before it is accepted, unless a consideration has been paid for the privilege of acceptance for a definite time. Where an offer has been made and no consideration paid to keep it open for a given time it is supposed to stand for what the law will consider a reasonable time.

The party making an offer has the right to prescribe in it the time, place, form, or other condition of acceptance, in which case such offer can be accepted only in the manner prescribed.

An offer may be accepted by merely acting upon it, the act becoming an acceptance from the time it was performed.

In order that a misrepresentation of facts may make a

contract invalid, it must have been made with a fraudulent intent. A non-disclosure of fact is equivalent to a misrepresentation of fact, provided the disclosure properly accompanied the transaction.

All contracts and agreements can in general be assigned by either party and the contract enforced by the assignee, except such contracts or agreements as involve a personal trust or confidence in one or both of the parties. Of this latter character are nearly all kinds of personal services. All building and engineering contracts are assignable, unless the writings themselves contain conditions denying such privileges. It is customary to insert a clause in building and engineering contracts making them non-assignable.

Lesson No. 56. Framing of Contracts

The first rule to be followed in the construction of contracts is to ascertain the real intention of the parties at the time the contract was signed. In fact, all rules are merged in this one, and have for their object the determination of the original real meaning of the document.

In arriving at this real meaning the words used must be understood in their ordinary and popular meaning when these do not have a technical significance. In all other cases the language is supposed to mean what it would ordinarily be understood to mean under the given circumstances of time and place and as between the given parties.

In general, any oral or written agreement may be altered at pleasure after it has been signed, when this is done by mutual consent.

In the case of engineering contracts, where it is common to have sureties or bondsmen, who guarantee faithful performance, such sureties must always be consulted and their consent obtained to any material change in the original contract which may be mutually agreed on by the principals.

In all kinds of engineering work changes in the original contracts are very common. Very seldom is an engineering

project completed strictly in accordance with the original plans and specifications, and contracts for such work should provide for changes, and in general changes should be made in accordance with such provisions.

In general, contracts may be discharged and the parties thereto freed from all further obligation in any of the following ways: 1, by agreement; 2, by performance; 3, by impossibility of performance; 4, by operation of law; 5, by breach.

A contract may contain a provision for its own discharge on the happening of some event or contingency. This contingency may be the nonfulfillment of some specific clause in the contract itself. Engineering contracts sometimes contain a clause to the effect that the work may be stopped at any time with a specified notice at the option of the party paying for the same.

Nearly all the works designed by engineers and architects are executed by other parties called contractors. The contractor usually buys all the materials and furnishes all the labor required in the execution of the work, and he agrees to do this within a stated time and for a fixed sum. To insure his doing this satisfactorily, certain written documents are prepared and signed by both parties. Standing between these two parties is the engineer or architect, who has planned the work, and who usually superintends its execution and assists in the final settlement between the parties to the agreement. Although paid by the party having the work done, he is expected to act justly and fairly toward both parties. In order that there shall be no misunderstanding in regard to the intentions of the designer, plans are usually drawn showing the general and detail features of the work, and accompanying these there are written descriptions of the work, of the material used, of the time and manner of payments, etc. This document is called the specifications. The drawings and this description are referred to as the *plans and specifications.*

To get open and general competition in the work of a particular contract, *bids* are advertised for, and if the undertaking is a large one, blank *forms of proposals* are prepared by the engineer, to be filled in by the bidders.

When bidders are called for it is often required that a fixed sum, known as a *guaranty*, be deposited with the bid to insure that the successful bidder — that is, the one whose bid is accepted — will sign the contract and meet the necessary conditions. This deposit is forfeited to the party letting the work in case the bidder fails or refuses to enter into a contract for the performance of the work after the award has been made to him. The deposits made by the unsuccessful bidders are, of course, returned to them, and that of the successful bidder is held until he has entered into the called-for contract. If there were no such guaranty required, the party receiving the award might after more careful investigation refuse to undertake the contract for the sum stated, and in this way cause the delay and expense of re-advertising the work. The advertiser usually states that he reserves the right to reject any or all bids, for if this is not done, the fair inference is that the contract will be let to the lowest bidder.

Lesson No. 57. Instruction to Bidders

It is impossible to make a fair and just bid upon any kind of work without the fullest knowledge of all the conditions and requirements involved. Note the following advertisement:

Office of the Mayor, Birmingham, April 9, 1899. — Sealed proposals will be received at this office until 12 o'clock noon, April 22, for building a six-foot sidewalk of two-inch pine plank, with 4 by 6 cedar runners resting on cedar ties, to extend on both sides of Main Street from Front Street to Hanover Street. Other necessary information may be obtained at this office.

Before a contractor should offer a bid upon a piece of work of this kind he should make the most careful estimate

(1) as to the actual cost of the material necessary, (2) as to the grading required, (3) as to the cost of labor, (4) as to the limit of time given for the completion of the contract. If these things have all been considered, and the sum of say $485 has been fixed upon as the lowest amount which will leave a reasonable amount of profit, the letter of proposal might properly take something like the following form:

April 20, 1899.

To the Mayor of Birmingham:

Sir, — The undersigned hereby proposes to build the sidewalks on both sides of Main Street as called for in your advertisement in the *Herald*, dated April 9, meeting all the conditions set forth, for the sum of Four Hundred and Eighty-Five Dollars ($485).

Respectfully Yours,

27 Main Street. James Brown.

Advertisements of large engineering and building contracts usually contain some such words as "plans, specifications, forms of proposal, and other information may be obtained upon application." In such cases it is necessary to make the proposal or bid upon the prescribed form prepared by the engineer or architect. As a general rule, the bidder should avoid including in his bid any suggestion or proposal not called for in the specifications. Where printed forms of proposal have been prepared by an architect or engineer, it is customary to reject all bids not made out on these forms, as well as all bids which, though made on the printed forms, have changed the conditions of the same in any particular, either by erasures, interlineations, or additional conditions. If the bidder considers it wise to submit a proposition in a different way, or with other conditions than those stated, he should submit his bid upon the printed form without correction or change, and then append to his bid an additional paper embodying such changes as he would wish to make, and the price he would make if these changes were agreed to.

Lesson No. 58. Specifications

Building and engineering *specifications* consist of a series of specific provisions, each one of which defines and fixes some one element of the contract. These clauses relate usually (1) to the work to be done, and (2) to the business relations of the two parties to the contract. A standard authority upon specifications classifies the important general clauses as follows:

1. Time of commencing the work.
2. Time of completing the work.
3. Monthly estimates upon which to base installment payments.
4. Provision for inquiring into the correctness of the monthly estimates.
5. Reserving a certain percentage as a repair fund for a stated period after completion.
6. Conditions of the final acceptance of the work.
7. Determination of damages sustained through failure to complete work within the prescribed time.
8. No claims for damages on account of suspension of work.
9. No claims for damages on account of delay.
10. Protection of property and workmen.
11. Protection against claims for the use of patents.
12. Assignment of contract.
13. Contractor not released by subcontractor.
14. Cancellation of contract through default of contractor.
15. Workmen's quarters and other temporary buildings.
16. Cleaning up after completion.
17. Provision for public traffic.
18. Cost of examination of completed work.
19. Contract subject to interpretation or change by architect or engineer.
20. Documents composing the contract.

The specifications will, of course, include the fullest par-

ticulars in detail of all the work to be done. When changes are introduced in the plans or specifications after the contract is let, such changes create a new contract, necessitating a new agreement as to compensation. Without a special clause authorizing such changes neither party could change the terms of the contract against the will of the other without breaking it: and furthermore, without some understanding as to how the compensation should be determined for such changes in plans or specifications, the owner would be at the mercy of the contractor; he could charge extravagant prices for the changes and there would be no remedy. It is customary to insert a clause providing for possible changes.

In describing the work to be done the following particulars should be noted: Describe the work first as a whole, then in detail; describe every portion in clear and simple language; make all statements of measurements, weights, etc., positive and exact; make the particulars regarding material so clear as to allow no doubt or unexpected choice; rigidly enforce the letter as well as the spirit of the contract.

Lesson No. 59. Building Records

It is a very common saying that the bookkeeping of architects and builders is the most slipshod and careless of any in the business world. It certainly should be quite as easy to prepare a plan for recording the debits and credits of building operations as to prepare plans and specifications for buildings costing hundreds of thousands of dollars. Almost any one conducting building or other similar operations should be able to frame a plan of bookkeeping adapted to his particular kind of work. The record, in any case, should be so clear and orderly that the contractor can get at a glance the fullest particulars of the cost or condition of any particular piece of work. All records should be immediately available in forming estimates for later bids.

NOTES, HINTS, AND ANSWERS

Lesson No. 1

1. *Answer:* $739.20. To find the total number of feet multiply 1760 by 3 and then by 20. Divide this by 50 to find the number of pieces each 50 ft. long. Then multiply by 35.

2. *Answer:* $291.65.

3. *Answer:* $4992. There would be 24 mi. of fence. Reduce to rods and multiply by 65.

4. *Answer:* 2673 in.

5. *Answer:* $6831. The water pipes will be exactly $1\frac{1}{8}$ mi. long.

Lesson No. 2

1. *Answer:* 528 yd.

2. *Answer:* 1364 yd.

3. *Answer:* 310 ch. Divide by 66.

4. *Answer:* $1,640,000. Divide by 80 to reduce chains to miles; then multiply by price per mile.

5. *Answer:* $10,944. There will be 24 mi. of fence necessary. Multiply this by 80, and we get the number of chains. Notice that the entire farm is to be cut up into 8 square fields, each with an area of 1 sq. mi.

Lesson No. 3

1. *Answer:* $27.44. Multiply 28 by 14 and then by 7.

2. *Answer:* 7,128 sq. in.

3. *Answer:* $20.25.

4. *Answer:* $183.15. There are $5\frac{1}{2}$ yd. in a rod, and $49\frac{1}{2}$ yd. in 9 rd.

5. *Answer:* $249.48. The fence is 126 rd. or 2079 ft. long. Multiply this by 24, and we have the number of

square feet to be painted. Divide by 9 to reduce to square yards; then multiply by $4\frac{1}{2}$.

6. *Answer:* $ 740.

7. *Answer:* 484 sq. yd.

8. *Answer:* $ 222,640. There are 1760 yd. in 1 mi. Multiply by $2\frac{1}{2}$ to get the length of the street in yards. One chain is 22 yd. The length and breadth in yards multiplied will give the number of square yards. Multiply this by the price.

9. *Answer:* 484 sq. yd.

10. *Answer:* $ 696.31. The total number of square yards is 1989. The walk is 90 yd. less 2 yd. at crossings long. This will give 88×2, or 176 sq. yd. in the walk. Then $1989 - 176$ will give the number of square yards of grass. The results are as follows:

Leveling	$ 59.67
Gravel walk	14.08
Sodding	217.56
Fencing	405.00
	$ 696.31

Lesson No. 4

1. *Answer:* 32 acres.

2. *Answer:* $ 649,600.

3. *Answer:* $ 979.20. One-half a mile is 40 ch. Multiply the width and we have 480 sq. ch. Divide by 10 to reduce to acres. Now 48 acres at 24 bu. to the acre and 85¢ a bushel will produce $ 979.20.

4. *Answer:* $ 3840. There will be 5 fences running each way. Each fence will be a mile long. This will make 10 miles, or 3200 rd., of fence.

5. *Answer:* $ 23,078.88. There are 3960 sq. yd. in the longer piece of street. The shorter is $90 - 22$, or 68 yd. long. This multiplied by 22 will give 1496 sq. yd. in the shorter piece. Add and multiply by 9 to reduce to square feet.

Lesson No. 5

1. *Answer:* 84 sq. ft.

2. *Answer:* \$ 42. The area is 1890 sq. ft., or 210 sq. yd.

3. *Answer:* 2880 acres. The diagram shows a square each side of which is 2 mi., and a triangle 2 mi. by $\frac{1}{2}$ mi. The area of the square is 2560 acres, and of the triangle 320 acres.

4. *Answer:* \$ 1500. The rectangular portion is 20 rd. by 6 rd. (33 yd.) and equals 120 sq. rd. The triangular portion is 18 rd. (132 yd. – 33 yd.) by 20 rd. and equals one-half of 18×20, or 180 sq. rd. The two together equal 300 sq. rd., or $1\frac{7}{8}$ acres.

5. *Answer:* There are 6400 acres on the smaller side and 16,640 acres on the larger. The course taken by the railway is considered a mere line.

Lesson No. 6

1. *Answer:* 9 cu. ft.

2. *Answer:* \$ 420.

3. *Answer:* 3520 loads. There are 5280 yd. in 3 mi. The road is 3 yd. wide and $\frac{8}{36}$, or $\frac{2}{9}$ of a yard deep. Multiply these three dimensions together, and we have the number of cubic yards or loads.

4. *Answer:* \$ 874.80. Divide the total number of cubic feet by 128 to get the number of cords.

5. *Answer:* \$ 176.80.

Lesson No. 7

1. *Answer:* 64 ft.

2. *Answer:* 696,960 stones. It will take 36 stones to cover a square yard of street. The depth of the stones need not be considered.

3. *Answer:* 20,000 lb., or 10 tons.

4. *Answer:* \$ 1540. Multiply the length, breadth, and depth, in yards, together to get the number of loads.

5. *Answer:* \$ 1800.

Lesson No. 8

1. *Answer:* 54 ft. Divide the total number of cubic inches by 144. Or by taking 12-ft. lumber as a basis: a board of this lumber 12 ft. long equals $18 + 18 = 36$ bd. ft.; a board 18 ft. long equals $36 + \frac{1}{2}(36)$, or 54 ft.

2. *Answer:* 648 ft. $(24 \times 12 \times 18 \times 18) \div 144$. This can easily be done by canceling. When the length is given in feet, as it nearly always is, multiply the length in feet by the breadth and thickness in inches, and divide by 12 to get the number of board feet.

3. *Answer:* \$ 9.

4. *Answer:* \$ 724.35.

5. *Answer:* \$ 230.40. The surface area of the platform is 4608 sq. ft. The lumber is 2 in. thick.

Lesson No. 9

1. *Answer:* 147 sq. ft. The average length of the parallel sides is $10\frac{1}{2}$ ft.; this multiplied by the altitude gives 147.

2. *Answer:* \$ 62.10. The average length of the parallel sides is one-half of $(28 + 18)$, or 23 ft.; this multiplied by the altitude or width will give the number of square feet.

3. *Answer:* \$ 4014.72. Reduce the lengths to feet. Suppose the blocks were shoved together so as to shut up Elm Street. We would then have a rectangular figure with 164 yd. or 492 ft. frontage on First and Second streets, and 272 ft. frontage on Maple and Pine streets. Then $492 \times 272 =$ 133,824 sq. ft., the size of both lots. This multiplied by the price will give the answer. The example can be worked also by the rule explained in this lesson. The method shown here is slightly shorter.

4. *Answer:* \$ 3536. The base is 60 ft. and the perpendicular distance is 272 ft. Then 60×272 will give the number of square feet. Reduce to square yards and multiply by \$ 1.95.

5. *Answer:* \$ 34. The area of *AFGB* is the average

length, or $4\frac{1}{4}$, multiplied by 2, or $8\frac{1}{2}$ sq. ft. Multiply this by the width and we have 17 cu. ft.

Lesson No. 10

1. *Answer:* $ 633.60. When the lumber is 12 foot the breadth in inches is the number of feet, board measure; therefore the length around the field in inches will be the number of feet of lumber in the fence.

2. *Answer:* $ 1503.36. If the walk were in a straight line it would be 1392 ft. long. Multiply by the width to get the number of square feet of surface. Each square foot is 2 ft. of lumber, because it is 2 in. thick.

3. *Answer:* $ 99. Each box has 48 ft. of surface; it will therefore take 48 ft. of lumber to make one box. The cost of the lumber will be $ 259.20. Add to this the $ 6 and the cost of making and you have the entire cost, $ 325.20.

NOTE. — No allowance is made for the saving of lumber at the ends and sides on such questions as these; the full surface measure is reckoned in every instance.

4. *Answer:* $ 596.25.

5. *Answer:* $ 445.50.

Lesson No. 11

1. *Answer:* 20,909 sods. Reduce the acre to square feet and divide by the number of square feet in one sod. Of course the fraction of a sod over in the division is considered as one sod.

2. *Answer:* 810 lots. There are 10 sq. ch. in each acre; there are $202\frac{1}{2}$ acre lots, or 810 quarter-acre lots.

3. *Answer:* $ 488.40.

4. *Answer:* $ 88.55.

5. *Answer:* $ 832. Find the number of square feet of surface and multiply by 4 (the thickness) to get the number of feet of lumber.

Lesson No. 12

1. *Answer:* \$ 66.49.
2. *Answer:* \$ 3.52.
3. *Answer:* \$ 12. The height of the ceiling does not signify.
4. *Answer:* \$ 40.53.
5. *Answer:* \$ 26.32.

Lesson No. 13

1. *Answer:* 96 perches. The number of cubic feet is 2376. This multiplied by 4 equals 9504. Divide by 100 and add 1% and you have $95.04 + .95 = 95.99$, or 96. The result can be found in this instance quite as easily by dividing 2376 by $24\frac{3}{4}$.

2. *Answer:* 107 perches. The entire length of the wall is 110 ft. The number of cubic feet is 2640. Multiply by 4, cut off two decimal places, and add 1%.

3. *Answer:* \$ 588. There are 140 perches.

4. *Answer:* 113 perches of stonework. The main wall equals 2160 cu. ft.; the central wall 446 cu. ft.; the short cross wall 193 cu. ft. Add another foot for the fraction in each instance.

5. *Answer:* 2091 perches. The length of the wall, outside measurement, is 1848 ft.

Lesson No. 14

1. *Answer:* 242 yd. 14 rd. = 77 yd. $77 \times 3\frac{1}{7} = 242$.

2. *Answer:* 75 mi. The straight drives measure 42 mi., the outer circle 22 mi., and inner circle 11 mi.

3. *Answer:* \$ 356.40. The fence is 1980 ft. long.

4. *Answer:* \$ 247.50. The roll around each pillar, if laid out flat, would be 15 ft. long by 44 in. wide.

5. *Answer:* 136 ft. Each hoop, when welded, is 33 ft. long. $(33 \times 4) + 4 = 136$.

Lesson No. 15

1. *Answer:* 1386 sq. in.

2. *Answer:* $115\frac{1}{2}$ sq. yd; \$ 311.85. The area of the

whole circle = 1386 sq. ft. The area of the base of the monument is $346\frac{1}{2}$ sq. ft. The difference, or the area of the walk, is $1039\frac{1}{2}$ sq. ft., or $115\frac{1}{2}$ sq. yd.

3. *Answer:* $27.10. The area of the floor is 616 sq. ft. Multiply by 2 for the thickness, and you have 1232 feet of lumber.

4. *Answer:* 560 yd. One mile = 1760 yd. This divided by $3\frac{1}{7}$ gives 560.

5. *Answer:* (*a*) 308 sq. in., or $\frac{1}{8}$ of circle.
(*b*) 224 sq. in. The area of a quarter of the circle is 616 sq. in. The area of the triangle is 392 sq. in. The difference equals the area of *D*.
(*c*) 616 sq. in, or $\frac{1}{4}$ of circle.

Lesson No. 16

1. *Answer:* $25.76. Exact measurement. No allowance made for openings.

2. *Answer:* 17,760 lath. There are 4320 sq. ft. in the walls and ends; 4800 sq. ft. in the ceilings; and 1536 sq. ft. in the gables.

3. *Answer:* $61.18. The surface to be plastered is practically 437 yd.

4. *Answer:* 6555 lath. Take 437 yd.

5. *Answer:* $1523.52. Each office contains 828 sq. ft. Each hall contains 2070 sq. ft. The entire number of square feet is 57,132.

Lesson No. 17

1. *Answer:* 6380 yards of wire. 10 acres = 48,400 sq. yd. Length of field = 48,400 ÷ 88, or 550 yd.

2. *Answer:* 21,121 ties. 12 miles = 21,120 yd. Add one tie for end.

3. *Answer:* 4.5094 miles. Find the number of square

yards of walk ($ 25,000 ÷ $ 2.10), and then divide by the width of the walk in yards, and you have the length in yards.

4. *Answer:* $ 31,600. The average length is 869 yd., and the perpendicular width is 22 yd.

5. *Answer:* $ 1530.90.

Lesson No. 18

1. *Answer:* 174,240 stones.

2. *Answer:* 14,000 ft. There are 28 ft. of lumber in each scantling.

3. *Answer:* $ 7040, considering 23,466$\frac{2}{3}$ loads. An estimater would call this 23,467 loads, or perhaps 23,500 loads.

4. *Answer:* $ 6400.

5. *Answer:* 132,000 loads. Find the area of the cross-section in yards, and multiply by the length (440) in yards.

Lesson No. 19

1. *Answer:* $ 396. The fence is 2640 ft. in circumference, that is, in length, and 10 ft. high.

2. *Answer:* $ 326.70.

3. *Answer:* 1850 ft.

4. *Answer:* 14 in. by 7$\frac{1}{2}$ in. by 9 in. It is possible to make a box of a different, but not so convenient a shape, as, for example, 10$\frac{1}{2}$ in. by 10 in. by 9 in.

5. *Answer:* $ 79.20. Each of the smaller circles is one-quarter of the larger circle in area, and one-half in circumference.

Lesson No. 20

1. *Answer:* 79 perches.

2. *Answer:* $ 62.92. Find the circumference in yards, multiply by the depth in yards, and this by 2 and 24, and you will have the number of square yards of surface in the

sides. Add to this the number of square yards inside and outside in the bottoms, and multiply by the price.

3. *Answer:* 146,455 tons (nearly). Find the area of the circle in feet, multiply by $1\frac{1}{2}$, and this by $57\frac{1}{2}$, and you will have the number of pounds. Divide by 2000.

4. *Answer:* $19.53.

5. *Answer:* 1232 ft.

6. *Answer:* 3168 pickets.

7. *Answer:* $4,428,000.

8. *Answer:* $45.

9. *Answer:* 4400 loads.

10. *Answer:* $27.50. The area of the painted section of a post equals the circumference (44 in.) multiplied by the height, or $(3\frac{2}{3} \times 7\frac{1}{2})$ ft.

Lesson No. 21

1. *Answer:* 20,736 gal.

2. *Answer:* 144,375 lb.

3. *Answer:* (*a*) 115,500 lb.; (*b*) 13,824 gal.

4. *Answer:* 21 in. deep.

5. *Answer:* 1176 gal. in each tank, or 3528 gal. This problem is a difficult one, involving the finding of the volume of what is known in mathematics as the frustum of a cone. The vats are cone shaped, and half their volume is filled. The rule is as follows: To the sum of the areas of the ends add the square root of their product, and multiply the sum thus found by one-third the height.

Lesson No. 22

1. *Answer:* 1240 in. Reduce 2170 ft. to inches by multiplying by 144; then divide by (14×12), and by $1\frac{1}{2}$.

2. *Answer:* $118.27. The total length of the outside walk if in a straight line would be 624 ft., and of the cross walks the length would be 296 ft.

3. *Answer:* 640 ft. Reduce to cubic inches, divide by 144, because there are 144 cu. in. in a foot of lumber, and deduct one-sixth.

4. *Answer:* $43.56.
5. *Answer:* $3.56.
6. *Answer:* 4992 ft.
7. *Answer:* 15 ft.
8. *Answer:* $11\frac{1}{9}$ ft.
9. *Answer:* $112.
10. *Answer:* $38.88. The veranda is 18 ft. wide and 54 ft. long.

Lesson No. 23

1. *Answer:* 17,920 acres.
2. *Answer:* 31,360 acres.
3. *Answer:* 63 ft. by 45 ft.
4. *Answer:* 3884+ yd. Note the diagram. The lower right-hand quarter represents a square mile, or a piece of land 80 ch. on each side. To find the distance from the center to *B*, square the two sides, add, and take one-half of the square root. There are 80 ch. in a mile. $80^2 + 80^2 = 12,800$. The square root of 12,800 equals about 113 (113.12). One-half of this, or $56\frac{1}{2}$ ch., equals the distance from the center to *B*. The whole distance is $176\frac{1}{2}$ ch. There are 22 yd. in a chain. The answer, therefore, is 3883 yd., or, allowing for the omitted decimal, 3884 yd.

NOTE. — For assistance in the solution of this question the student is referred to Lesson I. in Part II. of this work, "Mensuration for Beginners."

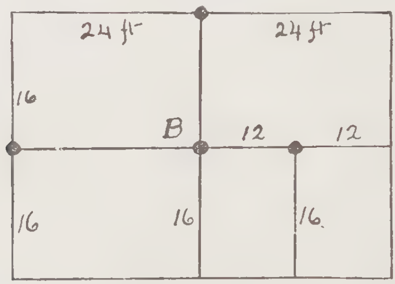
5. *Answer:* 20 mi. to the inch. A map $4\frac{1}{2}$ in. by 2 in. would on this scale represent a section of country 90 mi. by 40 mi.

Lesson No. 24

1. *Answer:* 12,500 shingles.
2. *Answer:* 39 bunches.
3. *Answer:* $1.41 gain.
4. *Answer:* 144 bunches. See Lesson No. 1, Part II.
5. *Answer:* 7320 ft.
6. *Answer:* $32.67.
7. *Answer:* $36.48.
8. *Answer:* $26\frac{3}{4}$ lb.
9. *Answer:* $60.
10. *Answer:* $14.

Lesson No. 25

1. *Answer:* $33.84. The walls and partition equal 256 ft. for each floor, or 6144 in. for the two floors.



Divide by 16, and we get 384 studding. Deduct 4 for each floor for the four points shown in the figure. If partitions cross at *B*, for instance, the same studding at the crossing point will answer for each partition. $384 - 8 = 376$. $376 \times 14 \div 700$ and multiplied by the wages = $33.84.

2. *Answer:* $17.76. There will be 148 joists needed, each 16 ft. in length.

3. *Answer:* $16.80.

4. *Answer:* 85 days.

5. *Answer:* $159.84.

Lesson No. 26

1. *Answer:* 11,232 bricks. Find the number of cubic inches in the pile and divide this by the number of cubic inches in a brick.

2. *Answer:* 16,212 bricks. There are 2316 sq. ft. in the wall, and each square foot required 7 bricks.

3. *Answer:* $447.55. Bricks, $288; lime, $3.75; sand, $1.80; labor, 40 d. at $3.85, $154.

4. *Answer:* $4698.04. There will be 519,120 bricks required. These will cost $2803.25. The lime will cost $259.56. The sand will cost $181.69. The labor will cost

$1453.54. In an actual estimate the number of brick would be taken as 520 M., or perhaps 525 M.

5. *Answer:* $1632. Consider the end of the wall at *A*. Take actual measurement inside and outside.

Lesson No. 27

Answers:

1. 25	7. 204	13. 23	19. 72	25. 145
2. 55	8. 239	14. 19	20. 81	26. 128
3. 74	9. 145	15. 27	21. 134	27. 24
4. 98	10. 278	16. 32	22. 84	28. 204
5. 109	11. 14	17. 75	23. 108	29. 105
6. 115	12. 15	18. 64	24. 132	30. 51

Lesson No. 28

1. *Answer:* $3\frac{3}{5}$ acres.
2. *Answer:* 6600 bricks.
3. *Answer:* $132.30.
4. *Answer:* 1200 sq. mi.
5. *Answer:* $63.65.
6. *Answer:* 427,680 stones.
7. *Answer:* $1810\frac{2}{7}$ sq. ft.
8. *Answer:* $22,376.91.

Lesson No. 29

1. *Answer:* 690 sq. yd.
2. *Answer:* 1026 sq. yd.
3. *Answer:* 20,883.54 + sq. yd.
4. *Answer:* 14,760 sq. yd.
5. *Answer:* 8077.68 + sq. yd.

Lesson No. 30

1. *Answer:* 70¢. There are 1120 sq. ft. of floor. $784.00 ÷ 1120 = 70¢.

2. *Answer:* 90¢. There are 2400 sq. ft. of floor.

3. *Answer:* $36,352.80. The total floor area is 38,880 sq. ft.

4. *Answer:* (*a*) $42,120. There are 780 sq. ft. in each floor. (*b*) 10¢. 1560 at 90¢ ÷ (1560 × 9) = 10.

5. *Answer:* $17,178.48. Add 10% to $13\frac{1}{2}$ and we have

14.85; add 20% of this to itself and we have 17.82. Then the estimate should be 17.82¢ per cubic foot.

Lesson No. 31

1. *Answer:* \$426,384. In each house there are 2268 sq. ft.

2. *Answer:* \$1342.80. First floor equals 7884 cu. ft.; second floor equals 7008 cu. ft. $119{,}136 \div 14{,}892 = 8$. The price per cubic foot is 8¢.

3. *Answer:* 25¢.

4. *Answer:* \$51,840.

5. *Answer:* \$5415.

Lesson No. 32

1. *Answer:* 336 sq. ft.

2. *Answer:* 14 sq. mi.

3. *Answer:* \$640.

4. *Answer:* \$906.05.

5. *Answer:* 1620 lots.

Lesson No. 33

1. *Answer:* \$617.40.

2. *Answer:* \$396.86. The total area is 4240 sq. ft.

3. *Answer:* 16.32 squares. The solution is as follows:

There are 544 sq. ft. (32×17) in the left side. Each of the other sections of the main roof is $16\frac{1}{2}$ ft. at top and $4\frac{1}{2}$ ft. at edge, or an average of $10\frac{1}{2}$ ft. This multiplied by 17 and by 2 gives 357 sq. ft. in the two right sections. Each side of the wing measures $27\frac{1}{2}$ ft. at top and $15\frac{1}{2}$ ft. at edge, or an average of $21\frac{1}{2}$ ft. This multiplied by 17 and by 2 gives 731 sq. ft. in the two sides of the wing.

$$731 + 357 + 544 = 1632$$
$$1632 \div 100 = 16.32$$

4. *Answer:* 84 rolls.

5. *Answer:* \$25.60.

Lesson No. 34

1. *Answer:* 2500 lb.

2. *Answer:* .798 in. in diameter. The area of a cross-

section of the rod would be a half inch. To find the diameter from the area divide the area by $\frac{22}{7}$ and double the square root; that is, do the very opposite to what you would do if you were finding the area from the diameter.

3. *Answer:* (*a*) 4420 lb. (*c*) 31,429 lb.
(*b*) 9944 lb. (*d*) 49,107 lb.

Find the area in each instance of the cross-section of the rod. A cross-section of 1-in. area carries safely 10,000 lb.

4. *Answer:* 1.635 in. in diameter. Area of the cross-section equals $21{,}000 \div 10{,}000 = 2.1$. From this the radius and the diameter can be found.

Lesson No. 35

1. *Answer:* 52 tons. $(4 \times 6 \times 6) \div 9$ multiplied by .325 equals weight in tons. Multiply by 10 for the 10 joists.

2. *Answer:* 3750 lb. 3. *Answer:* 1080 lb.

Lesson No. 36

1. *Answer:* $247\frac{1}{2}$ sq. ft.
2. *Answer:* $173\frac{4}{7}$ acres.
3. *Answer:* \$1906.75. The solution is as follows:

$$\frac{63 \times 45 \times 11}{14} \div 9 \text{ at } \$3.20 = \$792.00$$
$$= \text{cost of ellipse.}$$
$$\text{Whole area} = 475 \text{ sq. yd.}$$
$$\text{Area of ellipse} = 247\tfrac{1}{2} \text{ sq. yd.}$$
$$\text{Area of outside} = 227\tfrac{1}{2} \text{ sq. yd.}$$
$$\text{at } \$4.90 = \$1114.75.$$

Lesson No. 37

1. *Answer:* 1440 lb. $(480 \times 135) \div 45 = W.$
2. *Answer:* 126 lb. $(441 \times 4) \div 14 = W.$
3. *Answer:* 1760 lb. $(528 \times 10) \div 3 = W.$

4. *Answer:* 4 lb. $\left.\begin{array}{l}18 \times 15 = 270 \\ 24 \times \ \ 6 = 144\end{array}\right\}$ Dif. $= 126$ at left.
$15 \times 18 = 270$ at right.
$(270 - 126) \div 36 = P.$

5. *Answer:* 7200 lb. $(4800 \times 21) \div 14 = W.$

Lesson No. 38

1. *Answer:* 80 lb. $(400 \times 5) \div 25 = P.$
2. *Answer:* 6 in. $(18 \times R) = (3 \times 36 \text{ in.}).$
3. *Answer:* 12 lb. $(36 \times 2) \div 6 = W.$
4. *Answer:* 3000 lb. $(100 \times 5 \times 6) \div 1 = W.$
5. *Answer:* 108 lb. $(36 \times 3) \div 1 = W.$

Lesson No. 40

1. *Answer:* 30 lb. The power is to the weight as 1 to 16.
2. *Answer:* 20 lb. The power is to the weight as 1 to 64.
3. *Answer:* 12 lb. Draw a diagram; begin with the lowest pulley and find the tension on each cable. The first is $(160 + 8) \div 2$ or 84; the second is $(84 + 4) \div 2$ or 44; the third is $(44 + 2) \div 2$ or 23; the fourth is $(23 + 1) \div 2$ or 12.
4. *Answer:* 22,578 lb.

Lesson No. 41

1. *Answer:* 600 lb. The weight is double the number of blocks times the power.
2. *Answer:* 1200 lb.
3. *Answer:* 148 lb. The weight of 72 lb. will require a power of 12 lb., which will reduce the actual power of 160 lb. to 148 lb.
4. *Answer:* 20 lb.

Lesson No. 42

1. *Answer:* 20 lb. $P \times 8\frac{1}{2} = W \times \frac{1}{2}(8\frac{1}{2} - 8\frac{3}{8}).$
2. *Answer:* 56 lb. $P \times 16\frac{3}{4} = W \times \frac{1}{2}(16\frac{3}{4} - 15).$

3. *Answer:* 400,000 lb. $P \times 20 = W \times \frac{1}{2}(20 - 19.99).$
$$2000 = W \times .005.$$
$$W = 400{,}000.$$

4. *Answer:* 20 lb. $P \times 6\frac{7}{8} = W \times \frac{1}{2}(6\frac{7}{8} - 6\frac{3}{4}).$

Lesson No. 44

1. *Answer:* 800 lb.
2. *Answer:* 60 lb.
3. *Answer:* 120 lb.
4. *Answer:* $\frac{5}{33}$ of a ton. $160 \times 5 \div 5280.$
5. *Answer:* 144 lb.
6. *Answer:* $3833\frac{1}{3}$ lb. $50 \times 1 \div 30 = 1\frac{2}{3}$ tons. Add 10 lb. per ton, or 500 lb.
7. *Answer:* The power acting horizontally.

Lesson No. 45

1. *Answer:* $11{,}733\frac{1}{3}$ lb. Diameter $= 28$ in. $28 \times 3\frac{1}{7} = 88$, the circumference. $(88 \div \frac{3}{4}) \times 50 \times 2 = 11{,}733\frac{1}{3}$.
2. *Answer:* 26,400 lb.

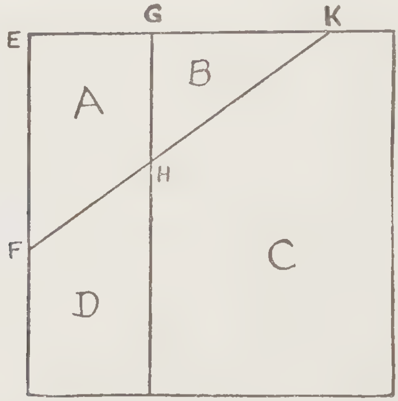
Lesson No. 46

1. *Answer:* 6000 lb.
2. *Answer:* 160 lb.
3. *Answer:* $9874\frac{2}{7}$ lb.

Lesson No. 47

1. *Answer:* 1080 acres.
2. *Answer:* \$159,720.
3. *Answer:* \$1658.06. The radius of the outer circle is $126 + 12$ or 138 ft. $= 46$ yd. The radius of the inner circle is 126 ft. $= 42$ yd. From the area of the outer semicircle subtract the area of the inner semicircle, and we have $553\frac{1}{7}$ sq. yd., the area of the circular part of the walk. The area of the straight part of the walk is 368 sq. yd.
4. *Answer:* The value of the lumber is \$696.96; the cost of painting is \$232.32. First obtain the diameter of the field in yards from the rule, "*To find the area of a circle, square the radius and multiply by* $\frac{22}{7}$."

5. *Answer:* $A = 5\frac{3}{5}$ sq. mi.; $B = 3\frac{3}{20}$ sq. mi.; $C = 20\frac{17}{20}$ sq. mi.; $D = 6\frac{2}{5}$ sq. mi. The solution of this problem is



quite simple when we know the distance from G of the point H. The whole triangle AB is similar in shape to the triangle B; therefore EK is to GK as EF is to GH. That is, $5 : 3 :: 3\frac{1}{2} : GH$. Therefore GH is $2\frac{1}{10}$ mi.

6. *Answer:* $1692\frac{17}{24}$.

Lesson No. 48

1. *Answer:* \$1209.60.

2. \$4.81. The average width is $3\frac{1}{2}$ ft.

3. *Answer:* 1008 ft. The area of a cross-section is 252 sq. in. Find the number of cubic inches and divide by 144.

4. *Answer:* \$1944.

5. *Answer:* \$913.50. The corners marked E measure 378 sq. ft.; the sections marked B measure 693 sq. ft.; the circles marked A measure 693 sq. ft.

6. *Answer:* \$164.80.

Lesson No. 49

1. *Answer:* 288 perches.

2. *Answer:* $69\frac{1}{11}$, or, in practice, 70 perches.

3. *Answer:* 1321 ties, allowing for one at each end.

4. *Answer:* $12\frac{69}{88}$, or 12.784 mi.

5. *Answer:* 15,092 ft.

6. *Answer:* 3465 loads. Find the area of the cross-section in feet; multiply by the length, or 66 ft.; divide by the number of cubic feet in a carload.

Lesson No. 50

1. *Answer:* 84,240 acres.

2. *Answer:* 100 ft. by 68 ft.

3. *Answer:* $5\frac{5}{9}$ ft., or 5 ft. $6\frac{2}{3}$ in.

4. *Answer:* $ 19.27.

5. *Answer:* $ 675.96.

6. *Answer:* 80¢.

7. *Answer:* 4015.4 tons.

Lesson No. 51

1. *Answer:* 50¢.

2. *Answer:* 32,310.74 + sq. ft.

3. *Answer:* $ 20,865.60. Each floor = 720 sq. ft.; 3 floors = 2160 sq. ft.; cost per square foot is (70¢ + 15%), or $80\frac{1}{2}$¢. Then $2160 \times 80\frac{1}{2} \times 12$ equals entire cost.

4. *Answer:* $ 268.27.

5. *Answer:* $ 890.68.

NOTE.—In the solution of this question the student must make use of the principles established in Lesson 3 of Part II. of this book, "Mensuration for Beginners."

6. *Answer:* $ 428.40.

7. *Answer:* 31 rolls.

8. *Answer:* 675 ft.

9. *Answer:* 29,040 sq. yd.

10. *Answer:* 88 yd.

Lesson No. 52

1. *Answer:* 5760 lb.

2. *Answer:* $8\frac{1}{3}$ lb.

3. *Answer:* 1375 lb.; not counting the weight of the pulley-block or friction.

4. *Answer:* $747\frac{2}{9}$ lb.

5. *Answer:* 100 lb.

6. *Answer:* $40\frac{1}{4}$ lb.

7. *Answer:* 300 lb.

Lesson No. 53

1. *Answer:* 2 mi. to the inch.

2. *Answer:* $10\frac{1}{2}$ ft.

3. *Answer:* $ 585.78.

4. *Answer:* 567 sq. mi.

5. $105.60. Considering both sides there are 880 sq. yd.

6. *Answer:* $61\frac{7}{8}$ ft.

7. *Answer:* $300.

8. *Answer:* $7700.

9. *Answer:* $77.22.

10. *Answer:* $648.

11. *Answer:* 48,000 cu. ft.

12. *Answer:* $34.44.

13. *Answer:* 14 in.

14. *Answer:* $67.95.

15. *Answer:* 432 cords.

16. *Answer:* 1320 posts.

17. *Answer:* 360 cu. ft.

18. *Answer:* $3671.95.

19. *Answer:* $2856.

20. *Answer:* 160 cu. yd.

II

MENSURATION FOR BEGINNERS

MENSURATION FOR BEGINNERS

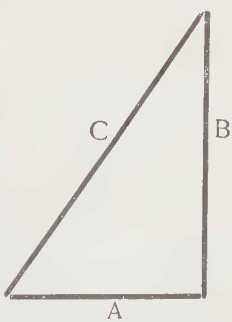
Lesson No. 1. Right-Angled Triangles

Mensuration tells us how to find the lengths of lines, the areas of surfaces, and the volumes of solid figures. **Geometry** and **trigonometry** establish certain rules and formulæ. **Mensuration** teaches their use and application.

When one straight line crosses another so as to form four equal angles, each of these angles is called a **right angle.** An angle smaller than a right angle is called an **acute angle,** and an angle larger than a right angle is called an **obtuse angle.**

A **triangle** is a plain figure formed by three straight lines. A **right-angled** triangle is a triangle which has a right angle. The three angles of a triangle if placed together will make two right angles.

The object of this lesson is to show how to find the length of a side of a right-angled triangle when the lengths of the other two sides are known.



A RIGHT-ANGLED TRIANGLE

$C =$ hypotenuse.

$A =$ base.

$B =$ perpendicular.

It is proved in geometry that the square of the hypotenuse is equal to the sum of the squares of the two sides.

(See Proposition 47 in "A First Course in Geometry.") That is,

$$C^2 = A^2 + B^2.$$

Now from this we get these formulæ:

$$C = \sqrt{A^2 + B^2}.$$

$$A = \sqrt{C^2 - B^2}.$$

$$B = \sqrt{C^2 - A^2}.$$

Now suppose that $A = 3$ in. and $B = 4$ in., then we have

$$C = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

$$A = \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3.$$

$$B = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4.$$

The **square** of a number is the product of the number multiplied by itself. The **square root** of a number is that smaller number which multiplied by itself will give that number. Thus, 25 is the square of 5; 4 is the square root of 16.

A subsequent lesson will show you how to find the square root of large numbers. In the exercises which follow, the square root will readily be seen. (See also Lesson 28 in "Mechanics' Bids and Estimates.")

EXERCISES

1. The sides forming the right angle of a right-angled triangle are 6 in. and 8 in. Find the length of the other side.

Note. — Square 6 and square 8; add together; take the square root of the sum.

2. The sides forming the right angle are 15 ft. and 8 ft. Find the other side.

3. The sides forming the right angle are 9 ft. and 12 ft. Find the other side.

4. The hypotenuse is 60 in. and the base 48 in. Find the perpendicular.

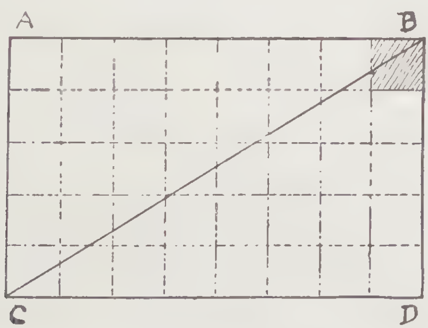
5. A ladder whose foot is placed on the ground 9 ft. from the front of a house reaches a window at a height of 12 ft. What is the length of the ladder?

6. A ladder 29 ft. long is placed so as to reach a point in the front of a house 21 ft. above the ground. How far is its foot from the house?

7. ABC is a triangle, and from A a perpendicular AD is drawn to the base BC. $AD = 12$ in., $BC = 25$ in., and $BD = 9$ in. Find the lengths of AB and AC.

Lesson No. 2. Rectangles and Squares

A **rectangle** is a four-sided figure containing four right angles: that is, it is a figure whose opposite sides are equal and parallel and whose angles are right angles. When the four sides of a rectangle are of equal length the figure is called a **square**.



The figure $ABCD$ is a rectangle. The line CB is called its **diagonal**. The diagonal divides the rectangle into two right-angled triangles.

To find the **area** of a rectangle we multiply the length by the breadth. If $ABCD$ is 8 in. long by 5 in. wide, it will

contain 8×5, or 40 sq. in. The proof of this is evident from the figure. If the figure is a square, we find the area by squaring one of the sides; that is, by multiplying it by itself.

If we know the area of a rectangle and the length of one of the sides, we can find the length of the other side by dividing the area by the length already known. If we know the area of a square, we can find the length of a side by taking the square root of the area.

EXERCISES

1. Find the area in square yards of a rectangle whose length is 93 ft. and whose breadth is 27 ft.

2. A rectangle is 36 in. long; it is one-fourth as wide as it is long. Find its area in inches.

3. The perimeter (the distance around) of a square is 28 ft. Find its area in square feet.

4. The diagonal of a rectangle is 15 ft. and the shorter side is 9 ft. Find the area.

5. The diagonal of a rectangle is 29 ft. and one of the sides is 20 ft. Find the area.

6. The area of a square is 169 sq. ft. Find its perimeter in feet.

7. How long will it take a man to walk around a square containing 40 acres, at the rate of 4 mi. an hour?

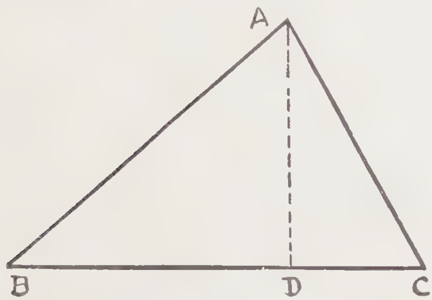
NOTE.—There are 160 sq. rd. in an acre, and a mile is 320 rd. long.

8. The area of a rectangle is equal to the sum of three squares whose sides are 18 ft., 19 ft., 20 ft. If one side of the rectangle is 31 ft., find the other side.

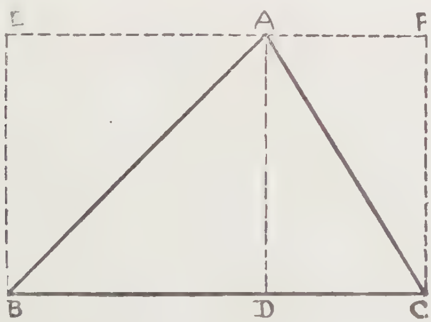
9. A rectangle is three times as long as it is wide. Its area is 243 sq. ft. Find its length in feet.

10. Find the length of a rectangle whose breadth is 9 ft. and whose area is equal to that of a square the diagonal of which is $\sqrt{288}$.

Lesson No. 3. Triangles



In the triangle ABC the perpendicular AD, drawn from A to the opposite side BC, is called the **altitude** of the triangle relative to BC as base.



To find the area of a triangle, having given one side and the perpendicular drawn to it from the opposite angle or vertex, we multiply the perpendicular, or altitude, by the given side, and take half the product.

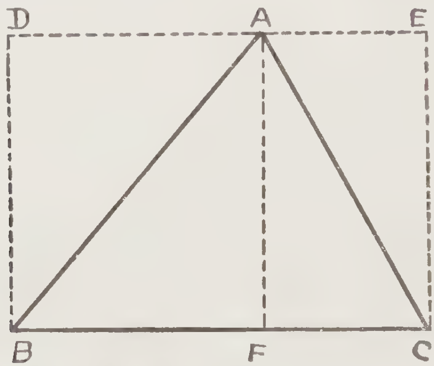
You will notice that the triangle ABD is one-half of the rectangle $EBDA$, and the triangle ADC is one-half of the rectangle $ADCF$, therefore the sum of these triangles or ABC is one-half of the sum of the rectangles or $EBCF$.

EXERCISES

1. Find the area in square feet of a right-angled triangle whose base is 9 ft. and altitude or perpendicular 12 ft.

2. The base of a triangle is 18 ft. and the altitude 12 ft. Find the area in square yards.

3. The two equal sides of an isosceles triangle (a triangle with two sides equal) is 42 in. and the altitude is 20 in. Find the area in square inches.



4. In the above figure AB is 15 ft., BF is 9 ft., and FC is 7 ft. Find the area of the triangle ABC.

5. The area of a triangle is 56 sq. ft. and the base is 16 ft. Find the altitude.

Lesson No. 4. Parallelograms

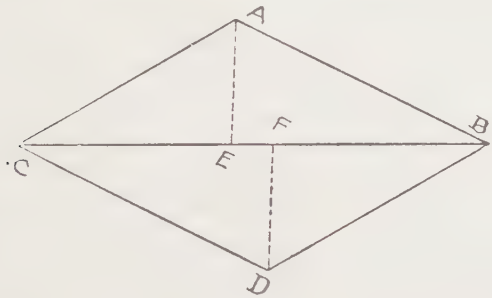
A **parallelogram** is a four-sided figure whose opposite sides are parallel.

It is shown in geometry

(1) *That the opposite sides of a parallelogram are equal.*
(2) *That the opposite angles are equal.*
(3) *That either diagonal divides the parallelogram equally.*
(4) *That the diagonals bisect each other.*

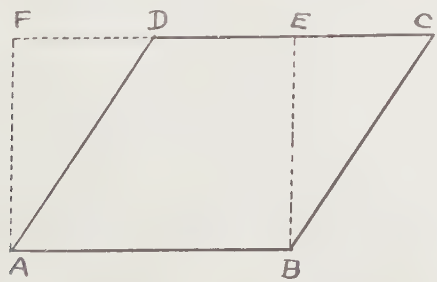
It is shown also that the perpendiculars drawn from one pair of opposite angles to the diagonal joining the other

pair are equal. Thus, if $ABCD$ is a parallelogram and CB a diagonal, the perpendicular AE is equal to the perpendicular FD.



To find the area of a parallelogram.

Let $DABC$ be the parallelogram. Then it is proven in geometry that parallelograms upon the same base and between the same parallels are equal; therefore the parallelo-



gram $DABC$ is equal to the parallelogram or rectangle $FABE$.

Hence the area of a parallelogram is equal to the area of a rectangle upon the same base and between the same parallels.

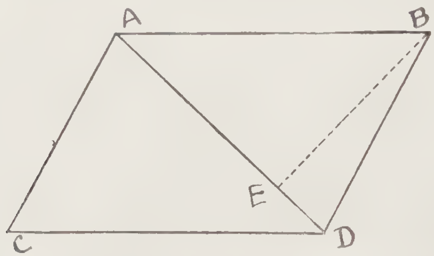
If a diagonal and a perpendicular upon it from one of the angles be given, the area can be found by multiplying

the diagonal by the perpendicular. (The student should prove for himself why this is so.)

NOTE. — In getting up this lesson the student should refer to Propositions 34 and 35 in "A First Course in Geometry."

EXERCISES

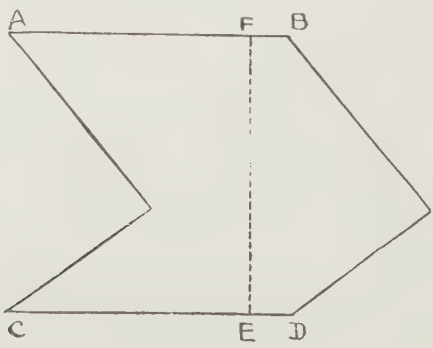
1. Find the area of a parallelogram the base of which is 12 ft. and the perpendicular height 9 ft.



2. Find the area of the parallelogram $ACDB$, having given that AD is 18 ft. and EB is 10 ft.

3. One side of a parallelogram is 960 yd. and the perpendicular distance between this and the opposite side is 605 yd. Give the area in acres.

4. The area of a parallelogram is 108 sq. ft. The perpendicular distance between the two sides is 9 ft. The perimeter is 44 ft. Find the length of the ends.



5. Find the area in acres of a piece of land of the shape of the above figure, having given that AB is 80 rd. and FE is 96 rd., and that the ends contain equal lengths and angles.

Lesson No. 5. Circles

The real value of the ratio which the circumference of a circle bears to its diameter cannot be exactly expressed in figures. To seven decimal places the value is 3.1415926.

For all practical purposes, and in the exercises which follow, consider this decimal as equal to $3\frac{1}{7}$.

The ratio of the circumference of a circle to its diameter is denoted in mathematics by the Greek letter π (pronounced *pi*). Thus we have

$$\frac{Circumference}{Diameter} = \pi.$$

Therefore the $Circumference = \pi \times Diameter.$

As above, for all practical purposes consider $\pi = 3\frac{1}{7}$.

ILLUSTRATIVE EXERCISE

Find the circumference of a circle whose radius is 161 in.

$$Circumference = \tfrac{22}{7} \times Diameter = \tfrac{22}{7} \times (161 \times 2) = 1012 \textit{ in.}$$

The area of a circle is found by multiplying the square of the radius by π. That is,

$$Area = (Radius)^2 \times \tfrac{22}{7}.$$

NOTE. — This is proven for us in trigonometry.

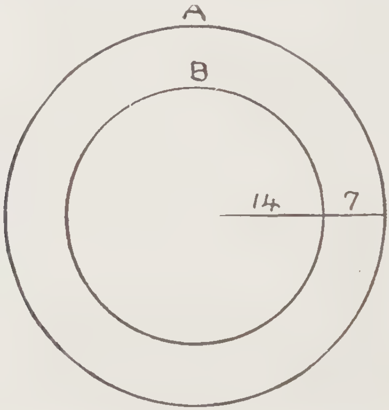
ILLUSTRATIVE EXERCISE

Find the area of a circle whose diameter is 28 ft.

$$Area = (Radius)^2 \times \tfrac{22}{7} = (14)^2 \times \tfrac{22}{7} = 196 \times \tfrac{22}{7} = 616 \text{ ft.}$$

Circles which have the same center are called *concentric* circles. The area of a plane circular ring inclosed between

two concentric circles is found by taking the difference between the areas of the two circles.



Thus, to find the area of the ring between the concentric circles A and B, find the area of A, and from this take the area of B.

EXERCISES

1. Find the length of the circumference of the circle, the radius being:

(*a*) 14 in.
(*b*) $3\frac{1}{2}$ ft.
(*c*) 1 ft. 9 in.
(*d*) 7 ch. 28 lk.
Note. — 100 links = 1 chain.
(*e*) 4 yd. 2 ft.

2. Find the radius of the circle, the circumference being:

(*a*) 264 in. (*b*) 73 yd. 1 ft. (*c*) 4048 in.

3. A wire may be bent into the form of a circle of radius 35 in. If the same wire were bent into the form of a square, what would be the length of its side?

4. A wire may be so bent as to inclose a square whose area is 121 sq. in. If the same wire were bent into the form of a circle, what would its radius be?

5. How far has a bicycle traveled when its driving wheel, 30 in. in diameter, has made 6300 revolutions?

6. A driving wheel of a locomotive engine, 7 ft. in diameter, makes 120 revolutions a minute. At what rate is the train traveling?

7. Find the area of the circles whose radii are respectively

(*a*) 7 in. (*c*) 4 yd. 2 ft.

(*b*) $3\frac{1}{2}$ in. (*d*) 1 ft. 9 in.

8. Find the radius of each circle, the area being given:

(*a*) 154 sq. in. (*b*) 3850 sq. ft. (*c*) 68 sq. yd. 4 sq. ft.

9. Find the radius of a circle equal in area to the sum of three circles whose radii are 8 in., 9 in., 12 in.

10. The radius of a circle is 5 in. What is the radius of another circle whose area is nine times that of the first?

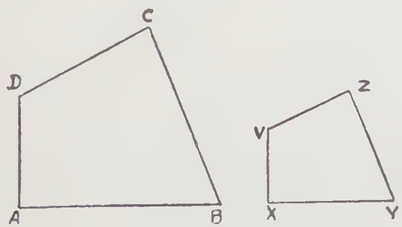
11. Find the area of a circle whose circumference is 22 in.

12. Find the circumference of a circle whose area is 616 sq. in.

Lesson No. 6. Similar Figures

Similar figures may be described as figures of the same **shape**, but not necessarily of the same **size**. Thus circles of all sizes are similar figures. Squares of all sizes are similar figures. The irregularly curved boundary of a country and its representation on a map are similar figures.

Rectilineal figures are similar, or of the same shape, when corresponding angles are equal and corresponding lines proportional.

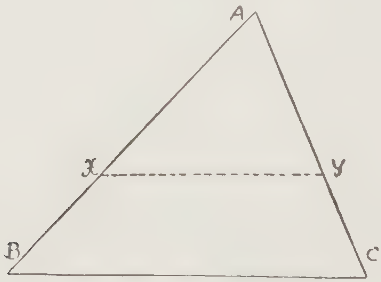


For instance, the two figures shown in the diagram are similar (1) if the angles at A, B, C, D are equal respectively to the angles at X, Y, Z, V, and (2) if

$$AB : XY :: BC : YZ :: CD : ZV :: DA : VX.$$

That is to say, if the side AB bears the same relation in length to XY that BC does to YZ, and that CD does to ZV, and that DA does to VX. In other words, if XY is one-half of AB, then each of the other sides of $VXZY$ must be one-half of the corresponding side of $DABC$.

Thus similarity in rectilineal figures includes two distinct properties. Two rectilineal figures of more than three sides might possess one of these properties and not the other, but they would not then be of the same shape. In the case of triangles it is proven in geometry that if one element of similarity exists, the other must necessarily exist also.



That is to say, if in two triangles ABC and DEF the angles at A, B, and C are respectively equal to the angles at D, E, and F, then the triangles are similar and the sides proportional; and if the sides are proportional the angles are equal. It is also shown in geometry that a straight line such as XY drawn parallel to the base BC of a triangle ABC, makes a triangle AXY in every way similar to the triangle ABC.

EXERCISES

1. The area of a square is 225 in. Find the area of a square whose side is one-third the length.

2. The area of a circle is 616 sq. in. Find the diameter of a circle one-quarter the area.

3. A map of a piece of land is drawn on a scale of 20 mi. to the inch. The map is $4\frac{1}{2}$ in. by 6 in. How many square miles of land?

4. There are two similar right-angled triangles. The sides of the larger are twice those of the smaller. If the area of the smaller triangle is 42 sq. in., what is the area of the larger triangle?

5. In the figure above, if AB is 24 in., BC 18 in., AC 16 in., and AX 15 in., find

(*a*) the length of XY; (*b*) the length of AY.

Lesson No. 7. Similar Figures (Continued)

This lesson continues the study of similar figures. It will be seen from the diagram that if we double the length and breadth of a rectangular figure we shall have a similar

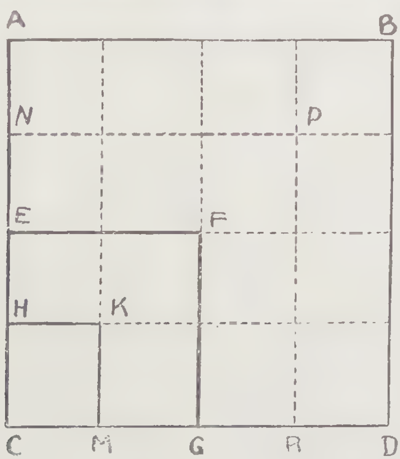
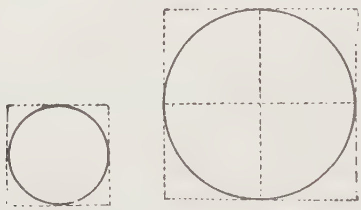


figure of four times the size. If we increase the length and breadth three times, as *HCMK* increased to *NCRP*, we shall have a figure nine times as large, and if we increase the length and breadth four times, we shall have a figure 16 times as large, and so on, the number of times as large being in each instance the square of the number of times as long or as wide.

Suppose that we have two circles, one struck with double the radius of the other, we say that the circumference of one is double that of the other, and that the area of one is four times that of the other. The circle is an inconvenient figure to illustrate this principle by, but by the circum-

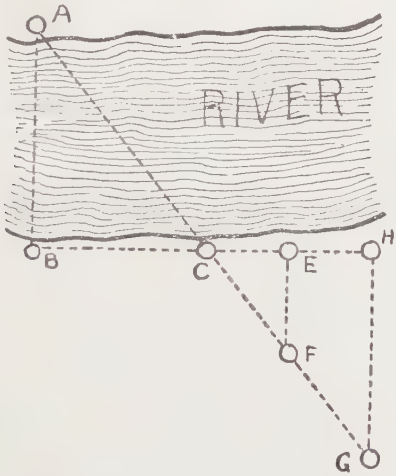


scribed squares it will be seen that the smaller circle is just one-quarter of the larger circle. If the diameter is three times as long, the circle will be nine times as large. In geometry this principle is proved accurately.

The importance of a knowledge of this principle consists in this, that it enables us to separate that part of the measurement which depends upon shape from that which depends upon size, and that when we have ascertained the former a variation in absolute size is dealt with as a mere question of proportion, depending upon the accurate measurements of some corresponding lines in each.

If, instead of giving the linear ratio between similar figures, we give the ratio of any surfaces or areas corresponding in each, the linear ratio will be the square root of the surface ratio.

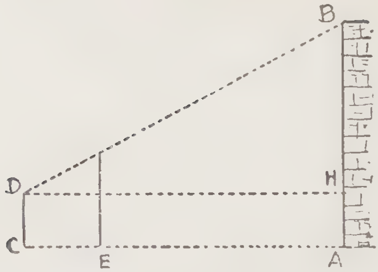
An application of the principle of similar figures may be seen in the accompanying illustration:



We desire to find the distance from B to A. We will suppose that A is a tree. We measure any distance BC along the shore and plant a stake at C. Then measure a distance CE, which shall be one-half or one-third, or some convenient fractional part of the length BC. Now move back at right angles to BE to a point F in a line with CA; that is, so that FCA shall be a straight line. Now the triangles ABC and CEF are similar, and knowing the sides of CEF and the length BC, we can easily find the length AB. If we make CH equal to BC, then HG will be the distance across the river, or AB.

EXERCISES

1. A rectangle is 4 in. by 5 in. Each side is increased to four times its present length. Find the area in square inches of the increased rectangle.



2. A man, wishing to ascertain the height of a tower, fixes a staff vertically in the ground at a distance of 27 ft. from the tower. Then retiring 3 ft. farther from the tower he sees the top of the staff in line with the top of the tower. If the observer's eye and the top of the staff are respectively 5 ft. 4 in. and 12 ft. above the ground, find the height of the tower. In the accompanying figure let AB represent the tower, EF the staff, and CD the observer in his final position.

3. The diameter of a circle is 14 in. Find the circumference of a circle having nine times the area.

4. In measuring the river in the foregoing figure, if CE is 24 ft. and BC is 72 ft., and it is found by actual measurement that EF is 32 ft., how wide is the river?

5. The area of a rectangle is 875 sq. in. Find the area of a similar rectangle whose sides and ends are one-fifth as long.

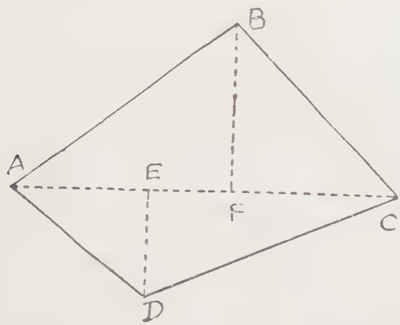
Lesson No. 8. Irregular Figures

Hitherto we have treated of plain figures bounded by either three or four straight lines, but a surface may be bounded by any number of lines. When all the lines and angles are equal, the figure is said to be **regular**. When any or all of the lines or angles are unequal, the figure is said to be **irregular**.

As in any rectilineal figure the number of angles is the same as the number of sides, sometimes the figure is named from its sides, sometimes from its angles.

Rectilineal figures may always be divided into triangles and quadrilaterals, whose areas can be separately found by the rules already learned. The sum of the results so obtained will be the area of the given figure.

A **trapezium** is a four-sided figure whose opposite sides are not parallel.



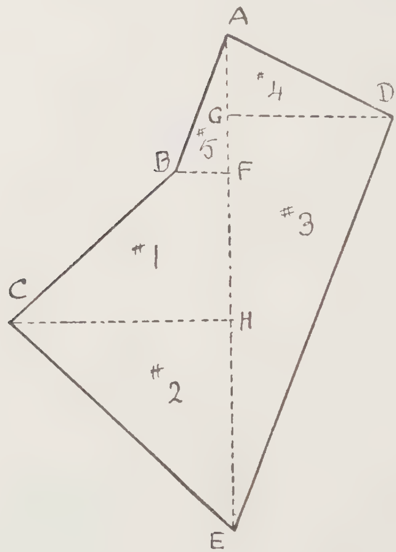
To find the area of a trapezium, divide it into two triangles and then find the sum of the areas of the triangles.

Thus the area of the trapezium $BADC$ is found by first making the measurements shown by the dotted lines, and then by finding the sum of the areas of the two triangles formed by running the diagonal AC.

It is not necessary to know the lengths of the sides. For instance, the area of ABC is found by multiplying one-half of AC by the altitude BF.

In practice irregular figures, such as that shown here, are usually subdivided by means of a base-line, as AE, and offsets from it. Suppose that this figure has the following measurements:

$AE = 80$ ch.	$EH = 40$ ch.
$CH = 40$ ch.	$HF = 25$ ch.
$BF = 12$ ch.	$AG = 10$ ch.
$DG = 36$ ch.	



To find the area in acres:

$$\text{Triangle No. } 2 = 40 \times 40 \div 2 = 800 \text{ sq. ch.}$$

$$\text{Triangle No. } 3 = 70 \times 36 \div 2 = 1260 \text{ sq. ch.}$$

$$\text{Triangle No. } 4 = 36 \times 10 \div 2 = 180 \text{ sq. ch.}$$

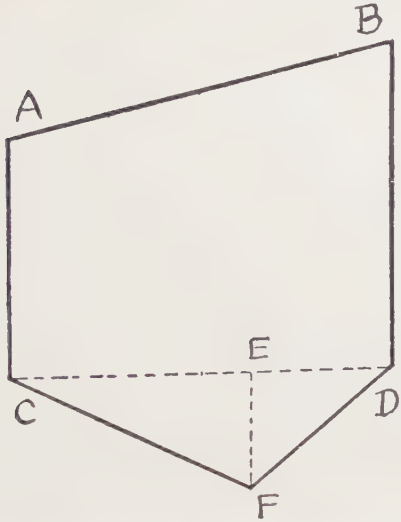
$$\text{Triangle No. } 5 = 12 \times 15 \div 2 = 90 \text{ sq. ch.}$$

$$\text{Trapezoid No. } 1 = 25(40 + 12) \div 2 = 650 \text{ sq. ch.}$$

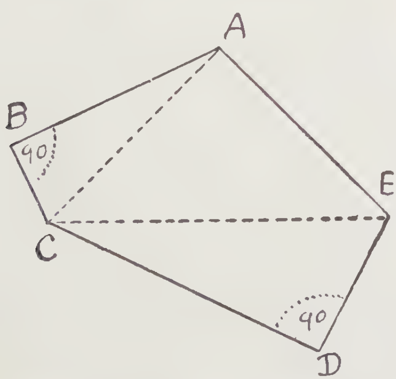
The total area is 2980 sq. ch., or 298 acres.

EXERCISES

1. The diagonal of a trapezium is 60 yd. and the perpendiculars are 40 yd. and 36 yd. Find the area in square yards.

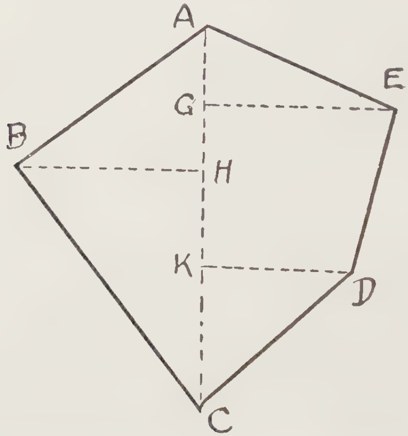


2. A field is the shape of the above diagram. AC is 20 rd.; BD is 30 rd.; CD is 28 rd.; EF is 16 rd.; ACD and BCD are right angles. How many acres in the field?



3. Calculate the area of the figure $ABCDE$ from the following data: The angles at B and D are right angles. $AB =$

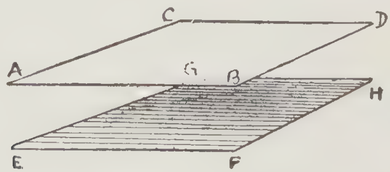
12 in.; $DE = 9$ in.; $BC = 5$ in.; $EA = 14$ in.; $CD = 12$ in. The perpendicular from A to CE is 11.2 in.



4. Calculate the area in acres of the figure $ABCDE$ from the following data: AC is 160 rd.; AG is 40 rd.; GK is 60 rd.; BH is 64 rd.; EG is 32 rd.; DK is 40 rd

Lesson No. 9. Rectangular Solids

We shall now begin the study of the measurements of solids. It is necessary to understand clearly the meaning of some terms which we shall have to employ. Note carefully the following definitions:



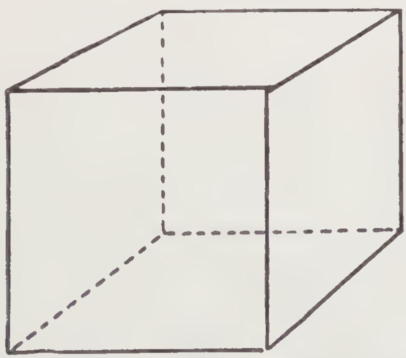
Parallel planes are such as do not meet one another, although produced. The plane $ABCD$ is parallel to the

plane *EFGH*. The floor and the ceiling of a room are parallel planes.

A straight line is said to be **perpendicular to a plane** when it makes right angles with every straight line which meets it in that plane. The student will readily understand when one plane is perpendicular or at right angles to another without a strict geometrical definition. Thus the walls of a room are perpendicular to the floor or to the ceiling; and the door while moving on its hinges remains perpendicular to the floor.

A solid figure, or **solid**, is that which has length, breadth, and thickness. A solid is bounded by one or more surfaces. If these surfaces are plane, they are called **faces**, and the bounding lines of the faces are called **edges**.

A **cube** is a solid bounded by six equal square faces.



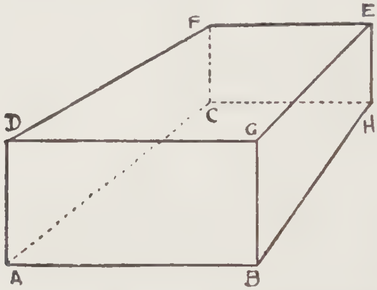
It will be seen from the figure that opposite faces of a cube are parallel, and that a cube has twelve equal edges.

A cube of which each edge is 1 in. in length is called a **cubic inch**. Similarly, if each edge measures 1 ft., or 1 yd., it is called a **cubic foot** or **cubic yard**.

The **volume**, or contents, of a solid figure is the space contained within its bounding surfaces.

A **rectangular solid** is a body bounded by six rectangular faces, the opposite faces being equal and parallel. This

figure represents a rectangular solid, of which AC is the length, AB the breadth, and AD the height. It is bounded by six rectangular faces, of which the opposite faces, as $ABGD$ and $CHEF$, are equal and parallel.



To find the whole surface of a rectangular solid, add together the areas of the six rectangular faces.

To find the volume of a rectangular solid, multiply together the three dimensions; that is, the length, breadth, and thickness.

If the dimensions are in inches, the product will represent cubic inches; if in feet, the product will represent cubic feet.

When the volume and two of the dimensions are given, the third dimension can easily be found by dividing the volume by the product of the given or known dimensions.

EXERCISES

1. Find in square feet the whole surface of a rectangular block of stone whose length is 2 yd. 2 ft., breadth 1 yd. 1 ft., and height 9 in.

2. How many yards of paper 22 in. wide are required for the walls of a room 15 ft. 4 in. long, 14 ft. 8 in. wide, and 11 ft. high?

3. The whole surface of a cube is 5 sq. ft. 6 sq. in. Find the length of each edge.

4. Find the height of a rectangular solid whose volume is 7 cu. ft. 864 cu. in., length 4 ft., and breadth 1 ft. 3 in.

5. Find the number of square feet in the surface of a rectangular solid whose length is 20 ft., breadth 15 ft., height 10 ft.

6. The sum of the edges of a cube is 288 in. Find the volume.

7. A certain book is 8 in. long, $5\frac{1}{4}$ in. wide, and $2\frac{1}{4}$ in. thick. How many such books can be packed in a box which is 3 ft. 6 in. long, 3 ft. broad, and 2 ft. deep?

Note. — Discover first whether they will pack into box without loss of space or not.

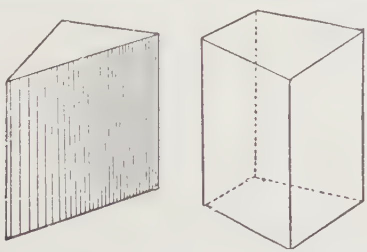
8. A reservoir is 24 ft. 8 in. long by 12 ft. 9 in. wide. Find how many cubic feet of water must be drawn off to make the surface sink 1 ft.

9. A piece of dressed timber is 2 ft. 9 in. by 1 ft. 7 in. at the end. What length will make 3 cu. ft.? (Give the answer in decimal form.)

10. The area of the floor of a square room is 225 sq. ft. and the total area of the four walls and ceiling is 765 sq. ft. Find the height of the ceiling.

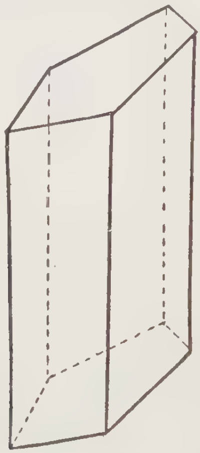
Lesson No. 10. Prisms

A **prism** is a solid whose sides are parallelograms and whose ends are similar equal and parallel plane figures.



If the sides of a prism are at right angles to its ends, as in the above illustration, it is a **rectangular** prism, and all its sides are rectangles.

If the sides of a prism are not at right angles to its ends, as in the illustration, it is called an **oblique** prism. In oblique prisms the sides are all parallelograms, but not all rectangles.



To find the solid contents or volume of a prism, multiply the area of one end by the perpendicular height.

In calculating volume we usually have the consideration of three dimensions; the product of two numbers, if the solid is wholly rectangular, gives us the area of the base, and the product of this and a third gives us the volume. This principle applies to the prism, for let us suppose that the area of each end of a prism is 1 sq. ft.; then it follows that if the prism is 1 ft. high its solidity is 1 cu. ft. If the prism be oblique, the area of one end must be multiplied by the perpendicular height. It is manifest that the volume increases as the perpendicular height increases.

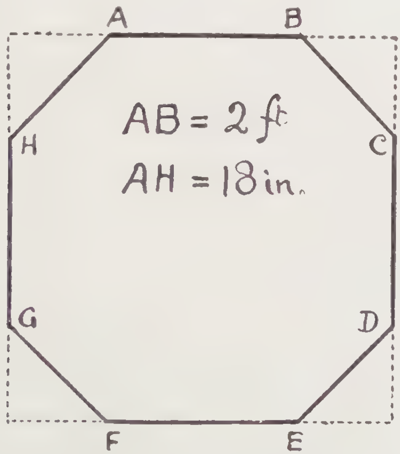
EXERCISES

1. The base of a prism is a right-angled triangle, the sides of which are 9 in., 12 in., and 15 in. The height is 12 ft. Find the volume in cubic feet.

2. The volume of a prism is a cubic foot, and the area of the base is 108 sq. in. Find the height.

3. Find the total surface of a prism, having given that the base is square, 17 in. each side, and the height 2 ft. 3 in.

4. The sides of the base of a triangular prism of stone measure 25, 20, and 15 in. respectively, and it is 10 ft. long. Find the total surface area of the block.



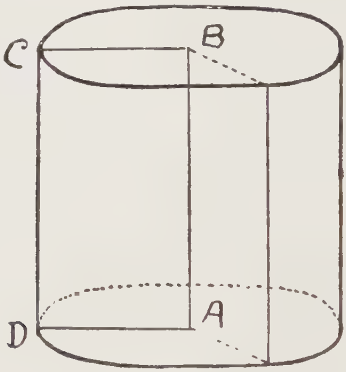
5. The base of a granite column is of the shape shown in the diagram. The height is 18 ft. Find the cost of polishing the sides of 20 such columns at 24¢ a square yard.

6. The base of a prism has four sides, two of which are parallel and 3 ft. apart. The parallel sides are $3\frac{1}{2}$ ft. and 18 in. respectively. The height is 9 ft. Find the volume in cubic yards.

Lesson No. 11. Cylinders

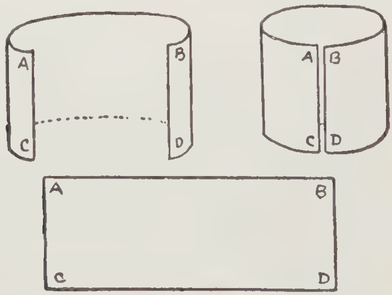
A **cylinder** may be described as a circular prism; that is, a prism with circular ends. The following is given as a mathematical definition:

A cylinder is a solid figure described by the revolution of a rectangle about one of its sides, which remains fixed.



Thus, if the rectangle *ABCD* revolves about the side *AB*, it describes the cylinder represented by the figure. *AB* is said to be the *axis* of the cylinder, and its height is the length of the axis.

The whole surface of the cylinder consists of the curved surface, described by *CD*, and the two circular ends, described by *AD* and *BC*. Either end, on which the cylinder may be supposed to rest, may be called its base.



It will be seen from these figures that if the surface of a cylinder were rolled out flat it would make a rectangle, as

ABCD, the length of which would be the circumference of the cylinder, and the breadth the height. To illustrate this, form a cylinder with a rectangular sheet of paper.

RULE 1: *To find the surface area of a cylinder: To twice the area of the base add the area of the rectangle formed by the circumference and the height.*

RULE 2: *To find the volume of a cylinder: Multiply the area of the base by the height.*

ILLUSTRATIVE EXERCISES

1. Find the surface area of a cylinder 14 in. in diameter and 20 in. high.

$$\begin{aligned} \textit{Circumference} &= 14 \times 22 \div 7 = 44 \text{ in.} \\ \textit{Area of curved surface} &= 44 \times 20 = 880 \text{ sq. in.} \\ \textit{Area of base} &= 7 \times 7 \times 22 \div 7 = 154 \text{ sq. in.} \\ \textit{Complete area} &= (154 + 154 + 880) \text{ sq. in.} \\ &= 1188 \text{ sq. in.} \end{aligned}$$

2. Find the volume in cubic inches of a cylinder 14 in. in diameter and 10 in. high.

$$\begin{aligned} \textit{Area of base} &= 7 \times 7 \times 22 \div 7 = 154 \text{ sq. in.} \\ \textit{Volume} &= (10 \times 154) \text{ cu. in.} \\ &= 1540 \text{ cu. in.} \end{aligned}$$

EXERCISES

1. Find the curved surface of a cylinder the diameter of which is 14 in. and height 7 in.

2. The diameter of a cylinder is 3 ft. 6 in. and the height 3 ft. 4 in. Find the entire area.

3. The radius of the base of a cylinder is 5 in., and its curved surface is 440 sq. in. Find its height.

4. The diameter of a cylindrical granite column is 21 in. and its height is 16 ft. Find the cost of polishing its curved surfaces at 36¢ a square foot.

5. How many square yards are covered in 90 revolutions of a cylindrical roller whose length is 4 ft. 6 in. and whose diameter is 3 ft. 6 in.?

6. Find the volume of a cylinder of which the radius of the base is 7 in. and the height 8 in.

7. Find (in pounds) the weight of a solid iron cylinder 1 ft. 9 in. long, the diameter of the base being 1 ft. 9 in.; supposing that one cubic foot of iron weighs $486\frac{3}{4}$ lb.

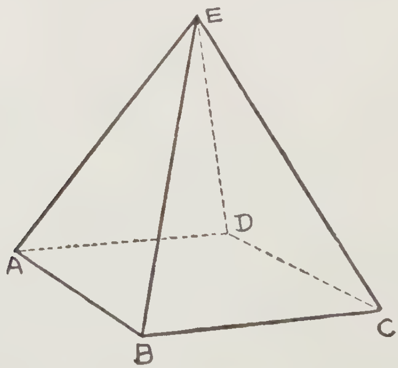
8. The volume of a cylinder is 308 cu. in. and the radius $3\frac{1}{2}$ in. Find the height.

9. The volume of a cylinder 14 in. long is equal to that of a cube having an edge of 11 in. Find the radius of the cylinder.

10. A circular well is 15 ft. deep and its diameter is 3 ft. 6 in. How many cubic yards of earth were taken out in digging it?

Lesson No. 12. Pyramids

A **pyramid** is a solid the base of which is any plane figure, and whose sides are triangles which meet in a point called the vertex of the pyramid.



The accompanying figure represents a pyramid, with base $ABCD$, and vertex E.

The base may be a triangle, a square, a rectangle, an octagon, or any regular or irregular plane figure contained by straight lines.

A pyramid is said to be **right** when a perpendicular dropped from the vertex on the base meets the base at its central point; that is, the center of its inscribed or circumscribed circle, if the base is a regular figure, or the intersection of its diagonals if the base is a rectangle.

The sum of the triangular faces is called the **slant** surface of the pyramid.

RULE 1: *The slant surface of a right pyramid equals one-half the perimeter of the base multiplied by the slant height.*

NOTE. — The whole surface equals the slant surface plus the area of the base.

RULE 2: *The volume of a right pyramid equals one-third the area of the base multiplied by the perpendicular height.*

NOTE. — In the exercises below all pyramids are supposed to be right pyramids.

EXERCISES

1. Find the slant surface of a pyramid 21 in. in perpendicular height, standing on a square base whose side is 40 in.

2. Find the volume in cubic inches of a pyramid whose perpendicular height is 21 in., standing on a square base whose side is 40 in.

3. A right pyramid, 3 ft. in perpendicular height, stands on a square base whose side is 8 ft. Find the area of one of the triangular surfaces.

4. Find the whole surface, including the base, of a right pyramid of which the perpendicular height is 2 ft., and the base a square on a side of 20 in.

5. Find the volume of a pyramid the base of which is a square on a side of 5 in., and the perpendicular height is 6 in.

6. The base of a pyramid is a rectangle measuring 8 in. by 4 in., and the height is 1 ft. Find the volume.

7. Find the height of a pyramid in which the volume is 84 cu. in., and the base a square on a side of 6 in.

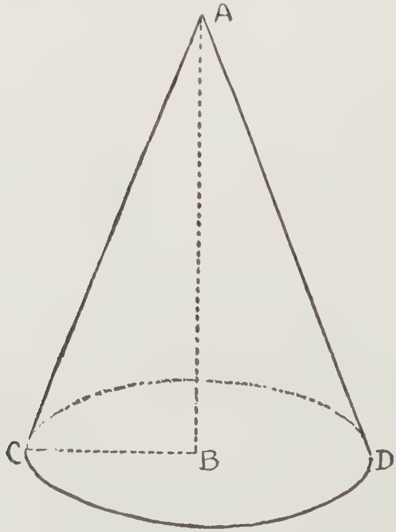
8. A pyramid, standing on a square base, is 15 in. in height. If the volume is 320 cu. in., find the side of the base.

9. The base of a pyramid is a right-angled triangle. The sides containing the right angle are 2 ft. and 5 ft.; the perpendicular height is 4 ft. Find the volume.

10. The height of a pyramid standing on a rectangular base is 2 ft. 8 in., and its volume is 5 cu. ft. If the length of the base is 3 ft. 8 in., find its breadth.

Lesson No. 13. Cones

A right circular cone is a solid described by the revolution of a right-angled triangle about one of its sides (one containing the right angle) which remains fixed.



Thus, if the right triangle ABC revolves about the side AB, it describes the cone represented in the figure. AB is said to be the **axis**, or height of the cone, and the angle CAD is called its **vertical angle**.

The whole **surface** of the cone consists of the curved surface described by AC, and of the circular end, or base, described by BC.

A cone is really a pyramid with a circular base, and the rules for finding the surface and volume of a cone correspond with those given for the pyramid.

RULE 1: *The slant surface of a cone equals one-half the circumference of the base multiplied by the slant height.*

NOTE. — The whole surface equals the slant surface plus the area of the base.

RULE 2: *The volume of a cone equals one-third the area of the base multiplied by the perpendicular height.*

NOTE. — In the exercises below all cones are supposed to be right circular cones.

EXERCISES

1. A cone is 30 ft. high, and the diameter of the base is 7 ft. Find its volume in cubic feet.

2. The slant height of a cone is 35 ft., and the diameter of the base is 42 ft. Find the entire surface area, including the base.

3. Find the curved surface of a cone of which the vertical height is 1 ft., and the radius of the base 2 ft. 11 in.

4. The curved surface of a cone is 396 sq. in., and the radius of the base is 6 in. Find its slant height.

5. The curved surface of a cone is 2 sq. ft. 42 sq. in., and its slant height is 1 ft. 3 in. Find the radius of the base.

6. A tent is cone-shaped. The perpendicular height is 12 ft., and the diameter 14 ft. Find the cost of the canvas at 60¢ a square yard.

7. How many square yards of canvas will be required for a conical tent 24 ft. high if the area of the base is 154 sq. ft.?

8. Find the volume in cubic inches of a cone, the height being 2 ft., and the slant height 25 in.

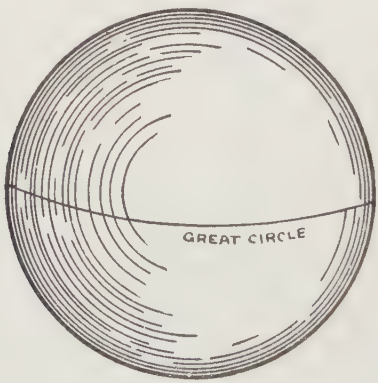
9. Find the weight of an iron cone the height of which is 14 in., and the diameter of the base 12 in., having given that a cubic foot of iron weighs 486 lb.

10. Find the volume of a cone having given that the area of the base is 1386 sq. in., and the slant height is 2 ft. 11 in.

Lesson No. 14. Spheres

A **sphere** is a body which is perfectly round in all directions, as a ball or globe.

Of course we are able to recognize a sphere at once from its shape, but, in order to prevent confusion, it is better, as in other figures, to associate the name with an accurate mathematical definition, such as the following:



A **sphere** is a solid described by the revolution of a semicircle about its diameter, which remains fixed.

A **sphere** may also be defined as a solid contained by one

curved surface, which is such that all points upon it are equidistant from a fixed point within, called the center.

When a plane intersects a sphere it always makes a circle, which is greater or smaller according as the intersection is near the center of the sphere or is more remote from it. When a plane passes through the center of a sphere the circle it makes is called a **great circle** of the sphere.

The **segment** of a sphere is any part of it cut off by a plane. If the plane passes through the center, it will divide the sphere into two equal parts called **hemispheres**.

For the working out of problems connected with spheres the following rules have been established by geometry:

1. *The surface of a sphere is equal in area to four times the area of a great circle of the sphere.*

2. *The surface of a sphere is equal in area to the area of a circle whose diameter is double the diameter of the sphere.*

3. *The surface of a sphere is equal in area to the area of the convex surface of the circumscribed cylinder.*

4. *The volume of a sphere is equal to the surface multiplied by one-third of the radius.*

5. *The volume of a sphere is equal to two-thirds of the volume of the circumscribing cylinder.*

Geometry has also established the following more general rules:

1. $\textit{Area of Surface} = 4 \times \frac{22}{7} \times (\textit{Radius})^2.$

2. $\textit{Volume} = \frac{4}{3} \times \frac{22}{7} \times (\textit{Radius})^3.$

NOTE. — The student should carefully commit these general rules to memory.

EXERCISES

1. Find the surface area of a sphere whose diameter is 7 in.

2. Find the volume of a sphere whose diameter is 7 in.

3. Find the surface area in square feet of a sphere whose radius is 4 yd. 2 ft.

4. A hollow cylinder is closed at the ends by hemispheres. If the length of the cylinder (without the ends) is 8 ft., and its diameter 6 ft., find the whole external surface in square feet.

5. What is the radius of a sphere whose surface is 154 sq. in.?

6. Find the volume of a sphere whose radius is 1 ft. 9 in.

7. Find the weight, in pounds, of a solid sphere of lead 14 in. in diameter, having given that a cubic inch of lead weighs $6\frac{1}{2}$ oz.

8. How many spherical bullets, each 1 in. in diameter, can be molded from a rectangular block of lead 11 in. long, 8 in. wide, and 5 in. thick?

9. How many solid spheres, each 6 in. in diameter, can be molded from a solid metal cylinder whose length is 45 in. and diameter 4 in.?

10. If a solid cylinder of lead 8 in. long and 6 in. in diameter weighs 92 lb., find the weight of a leaden sphere 1 ft. in diameter.

11. Find the radius of a sphere whose volume is $113\frac{1}{7}$ cu. in.

12. Find the volume of a sphere whose area is 616 sq. in.

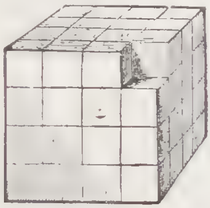
Lesson No. 15. Similar Solids

Similar solids may be described as solids having the same shape, but not necessarily the same size. All cubes are similar solids; all spheres are similar.

The following rules apply to all similar solids:

1. *The surfaces of similar solids are proportional to the squares of corresponding edges, or of any corresponding lines which may be drawn in them.*

2. *The volumes of similar solids are proportional to the cubes of corresponding edges, or of any corresponding lines that may be drawn in them.*



Suppose that the above illustration represents a cubical pile of cubes, each of which is 3 in. square; that is, on each edge. The outside area of a single small cube bears the same relation to the outside surface area of the whole cube that the square of 3 does to the square of 12 (the edge of the large cube).

If $\quad A =$ surface area of large cube

and $\quad B =$ surface area of small cube,

then $\quad A : B :: 144 : 9.$

But we know the surface area of B to be $(3 \times 3) \times 6 = 54$ sq. in.; then $(54 \times 144) = A \times 9$, and $A = (54 \times 144) \div 9 =$ 864 sq. in.

If $\quad C =$ volume of large cube

and $\quad D =$ volume of small cube,

then $\quad C : D :: (12)^3 : (3)^3.$

But we know the volume of the small cube to be 27 cu. in.; then $(1728 \times 27) = C \times 27$, and $C = (1728 \times 27) \div 27 =$ 1728 cu. in.

Note. — We can see from an examination of the illustration that the surface area of the large cube must be 4×4 times the surface area of the small cube; that is, 16 times 54 sq. in., or 864 sq. in. Also that the volume of the large cube must be $4 \times 4 \times 4$ times the volume of the small cube; that is, 64 times 27 cu. in., or 1728 cu. in. That is, in each case, observation shows that the principle of our rules is true.

EXERCISES

1. A rectangular solid is 12 in. by 8 in. by 6 in. What are the dimensions of a similar solid 27 times as large?

2. The diagonal of a rectangular solid is 2 in. The diagonal of another similar solid is 10 in. The surface area of the larger solid is how many times the surface area of the smaller solid?

3. The diameter of a sphere is 12 in., and its weight 96 lb. Find the weight of a sphere of the same material whose diameter is 30 in.

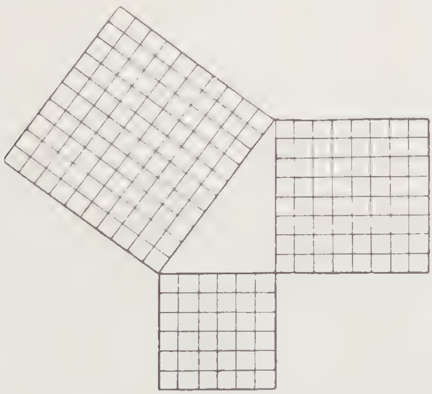
4. There are two cubes of the same material. The surface area of the larger is nine times that of the smaller. If the smaller cube weighs 2 lb., what is the weight of the larger?

5. A rectangular solid is 9 in. long. How much greater is the surface area of a similar solid 6 ft. long? How much greater is the volume?

NOTES, HINTS, AND ANSWERS

Lesson No. 1

1. *Answer:* 10 in. The square of the base is 6^2, or 36; the square of the perpendicular is 8^2, or 64. Now the sum of these is 100, which is equal to the square of the hypotenuse. Therefore the square root of 100, or 10, will be the length of the hypotenuse. Notice the proof of this in the following figure.



2. *Answer:* 17 ft. $15^2 + 8^2 = 289$; $\sqrt{289} = 17$.

3. *Answer:* 15 ft. $9^2 + 12^2 = 225$; $\sqrt{225} = 15$.

4. *Answer:* 36 in. $60^2 - 48^2 = 1296$; $\sqrt{1296} = 36$.

5. *Answer:* 15 ft. This is a simple right-angled triangle, with a base of 9 ft. and a perpendicular of 12 ft.

6. *Answer:* 20 ft. The hypotenuse and perpendicular are given; it is required to find the base.

$$29^2 - 21^2 = 400;\ \sqrt{400} = 20.$$

7. *Answer:* $AB = 15$ in. $AC = 20$ in. Draw the figure. You have two right-angled triangles. The perpendicular in each instance is 12 in. The base of one is 9 in. and of the other 16 in.

Lesson No. 2

1. *Answer:* 279 sq. yd. 31 yd. $\times$ 9 yd.

2. *Answer:* 324 sq. in. 36 in. $\times$ 9 in.

3. *Answer:* 49 sq. ft. 7 ft. $\times$ 7 ft.

4. *Answer:* 108 sq. ft. Square the diagonal (hypotenuse) and square the known side, subtract, and take the square root, and we have the other side. $\sqrt{15^2 - 9^2} = \sqrt{144} = 12$.

5. *Answer:* 420 sq. ft. Similar to No. 4.

6. *Answer:* 52 ft. The square root of 169 is 13. This is the length of one side. The perimeter is the sum of all the sides.

7. *Answer:* 15 min. 40 acres = 6400 rd. The square root of 6400 is 80, the length of one side. The four sides equal 320 rd., or 1 mi.

8. *Answer:* 35 ft. $19^2 + 18^2 + 20^2 = 1085$.

9. *Answer:* 27 ft. The rectangle equals three squares. The area of each square is $243 \div 3$, or 81 sq. ft. The side of each square equals the square root of 81, or 9.

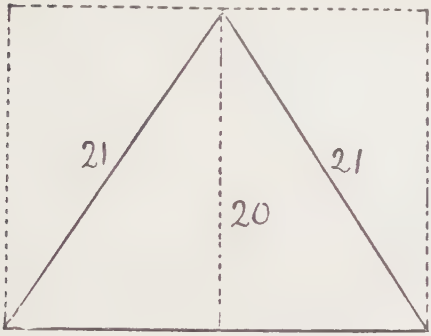
10. *Answer:* 16 ft. If diagonal equals $\sqrt{288}$ ft., the square of $\sqrt{288}$, or 288, equals the sum of the squares of two sides. Then the square of one side equals $288 \div 2$, or 144. The side of the square then equals the $\sqrt{144}$, or 12. The area of the square will equal 12×12, or 144. Then 144 equals the area of the rectangle, one side of which is 9.

Lesson No. 3

1. *Answer:* 54 sq. ft. $(9 \times 12) \div 2 = 54$.

2. *Answer:* 12 sq. yd. $(18 \times 12) \div 2 \div 9 = 12$.

3. *Answer:* 210 sq. in. $(21 \times 20) \div 2$. See figure.



4. *Answer:* 96 sq. ft. 15^2 equals 225; 9^2 equals 81; $225 - 81 = 144$. The square root of 144 is 12, or the altitude AF. The base BC equals $9 + 7$, or 16 ft. $(16 \times 12) \div 2 = 96$.

5. *Answer:* 7 ft. 56 divided by one-half the base.

Lesson No. 4

1. *Answer:* 108 sq. ft. $12 \times 9 = 108$.

2. *Answer:* 180 sq. ft. $18 \times 10 = 180$.

3. *Answer:* 120 acres. $(960 \times 605) \div 4840$, the number of square yards in an acre, gives 120.

4. *Answer:* 10 ft. each. $108 \div 9 = 12$, one of the sides. Both sides equal 24. $44 - 24 = 20$, the sum of the two ends.

5. *Answer:* 48 acres. The figure consists of two parallelograms. Their parallel sides are 80 rd. each, and the sum of their perpendiculars is 96 rd.; therefore the area in acres equals 96×80 divided by 160, 160 the number of square rods in an acre.

Lesson No. 5

1. *Answer:* (*a*) 88 in. (*b*) 22 ft. (*c*) 11 ft. (*d*) 45.76 ch. (*e*) 88 ft.

2. *Answer:* (*a*) 42 in. (*b*) 35 ft. (*c*) 644 in.

3. *Answer:* 55 in. The circumference of the circle would be 220 in. This would be the perimeter of the square; therefore one side would be one-quarter of 220.

4. *Answer:* 7 in. The side of the square would be 11 in., and four sides, or the circumference of the circle, would be 44 in.; therefore the radius would be one-half of $(44 \div 3\frac{1}{7})$.

5. *Answer:* 9.375 mi.

6. *Answer:* 30 mi. an hour. The circumference of the drive wheel is $7 \times 3\frac{1}{7}$, or 22 ft. $22 \times 120 = 2640$, or the number of feet the train moves per minute. This multiplied by 60 equals the number of feet per hour. Divide by 5280 to reduce to miles.

7. *Answer:* (*a*) 154 sq. in. (*b*) $38\frac{1}{2}$ sq. in. (*c*) $68\frac{4}{9}$ sq. yd. (*d*) $9\frac{5}{8}$ sq. ft.

8. *Answer:* (*a*) 7 in. (*b*) 35 ft. The area divided by $3\frac{1}{7} = 1225 =$ the square of the radius; then the square root of 1225, or 35, equals the radius. (*c*) 14 ft.

9. *Answer:* 17 in.

10. *Answer:* 15 in. The area of the smaller circle multiplied by 9 equals the area of the larger circle, and this divided by $3\frac{1}{7}$ gives the square of the radius, or 225 sq. in.

11. *Answer:* $38\frac{1}{2}$ sq. in. The diameter is $22 \div 3\frac{1}{7}$, or 7 in. The radius is $3\frac{1}{2}$ in. This squared and multiplied by $3\frac{1}{7}$ equals the area.

12. *Answer:* 88 in. The area divided by $3\frac{1}{7}$ gives the square of the radius.

Lesson No. 6

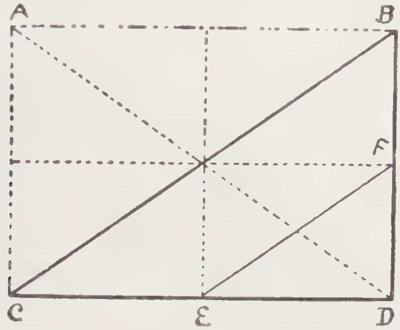
1. *Answer:* 25 sq. in. The square root of 225 is 15, one side of the first square.

2. *Answer:* 14 in.

3. *Answer:* 10,800 sq. mi.

4. *Answer:* 168 sq. in. Note the figure. If *FDE* is the smaller triangle, then *BDC* will equal the larger, and no matter what the sides of the triangles may be, the larger

triangle will always be four times the area of the smaller. Note also that EF is always parallel to CB.



5. *Answer:* $XY = 11\frac{1}{4}$ and $AY = 10$. This is simply a question of equality of ratios. $AB : AX :: BC : XY$; that is, $24 : 15 :: 18 : x$; or $15 \times 18 \div 24 = x$.

Lesson No. 7

1. *Answer:* 320 sq. in.

2. *Answer:* 72 ft. The staff is $6\frac{2}{3}$ ft. higher than the observer's eye. Then $3 : 6\frac{2}{3} :: (3 + 27)$: height of staff above vertical line of vision. $6\frac{2}{3} \times 30 \div 3 = 66\frac{2}{3}$. Add $5\frac{1}{3}$, the distance of the observer's eye from the ground.

3. *Answer:* 132 in.

4. *Answer:* 96 ft.

5. *Answer:* 35 sq. in.

Lesson No. 8

1. *Answer:* 2280 sq. yd. One-half the diagonal multiplied by the sum of the perpendiculars.

2. *Answer:* 5.775 acres. The area of $ACDB$ is found by multiplying CD by the average height, which is 25. The area of CFD is found by multiplying one-half CD by EF.

3. *Answer:* 168 sq. in.

4. *Answer:* 57 acres. $ABC = 5120$ sq. rd.; $AGE = 640$ sq. rd.; $DKC = 1200$ sq. rd.; $GKDE = 2160$ sq. rd.

Lesson No. 9

1. *Answer:* 82 sq. ft.

2. *Answer:* 120 yd.

3. *Answer:* 11 in. 5 sq. ft. 6 sq. in. = 726 sq. in.

$$726 \div 6 = \text{area of each side} = 121 \text{ sq. in.}$$

The square root of this gives the edge.

4. *Answer:* 18 in.

5. *Answer:* 1300 sq. ft.

6. *Answer:* 8 cu. ft. $288 \div 12 = 24$, the length in inches of each of the 12 edges.

7. *Answer:* 384 books.

8. *Answer:* $314\frac{1}{2}$ cu. ft. Simply the entire surface 1 ft. deep.

9. *Answer:* 8.2679 in.

10. *Answer:* 9 ft. The square root of 225, or 15, is the length of each side. From 765 take 225, and we have the area of the four walls. This divided by the length of the four walls, or 60, will give the height.

Lesson No. 10

1. *Answer:* $4\frac{1}{2}$ cu. ft. Area base equals $(12 \times 9) \div 2 = 54$ sq. in. $= (54 \div 144)$ sq. ft. $(54 \div 144) \times \text{height} = \text{answer}$.

2. *Answer:* 16 in. 1 cu. ft. = 1728 cu. in. $1728 \div 108 = 16$.

3. *Answer:* 16 sq. ft. 110 sq. in. The area of the ends is $2 \times (17 \times 17) = 578$ sq. in. Sides $= 4 \times (17 \times 27) = 1836$ sq. in.

4. *Answer:* 52 sq. ft. 12 sq. in. The area of the triangle at either end is 150 sq. in. The triangles are right-angled, as may be seen by examining the relation of their sides. The area of the sides of the prism is $(3000 + 2400 + 1800)$, or 7200 sq. in. Add the area of the ends, and we have 7500 sq. in., or 52 sq. ft. 12 sq. in.

5. *Answer:* $134.40.

6. *Answer:* $2\frac{1}{2}$ cu. yd. The average length of the 2 parallel sides is $2\frac{1}{2}$ ft. $2\frac{1}{2} \times 3 = 7\frac{1}{2}$ sq. ft. = area of base. $7\frac{1}{2} \times 9 = 67\frac{1}{2}$ cu. ft. $= 2\frac{1}{2}$ cu. yd.

Lesson No. 11

1. *Answer:* 308 sq. in.

2. *Answer:* 8052 sq. in. The area of the curved surface is 5280 sq. in. and of the ends 2772 sq. in.

3. *Answer:* 14 in. Diam. $= 10$. Circum. $= 10 \times 22 \div 7$. Area of curved surface, or 440, $\div$ circum. $=$ height.

4. *Answer:* \$ 31.68. Circum. $\times$ height $=$ area.

5. *Answer:* 495 sq. yd. Area of roller $= 11 \times 4\frac{1}{2}$, or $49\frac{1}{2}$ sq. ft. Multiply by 90 and divide by 9.

6. *Answer:* 1232 cu. in. Find the area of the base in square inches and multiply by the height in inches.

7. *Answer:* 2049.67 + lb. Find the number of cubic inches, reduce to cubic feet, and multiply by $486\frac{3}{4}$.

8. *Answer:* 8 in. Area of end $= 77 \div 2$, or $38\frac{1}{2}$.

$$308 \div 38\tfrac{1}{2} = 8.$$

9. *Answer:* $5\frac{1}{2}$ in. Cube $= 1331$ cu. in. Divide by 14, and we have the area of one end. Divide by $\frac{22}{7}$, and we have the square of the radius, or 30.25. The square root of this is 5.5.

10. *Answer:* 5.347 cu. yd.

Lesson No. 12

1. *Answer:* 2320 sq. in. The slant height will be found to be 29 in.

2. *Answer:* 11,200 cu. in.

3. *Answer:* 20 sq. ft.

4. *Answer:* 10 sq. ft.

5. *Answer:* 50 cu. in.

6. *Answer:* 128 cu. in.

7. *Answer:* 7 in.

8. *Answer:* 8 in.

9. *Answer:* $6\frac{2}{3}$ cu. ft.

10. *Answer:* $18\frac{9}{22}$ in.

Lesson No. 13

1. *Answer:* 385 cu. ft. Diameter $= 7$ ft.; radius $= 3\frac{1}{2}$ ft. $3\frac{1}{2} \times 3\frac{1}{2} \times 3\frac{1}{7} = 38\frac{1}{2}$, area of base. $(38\frac{1}{2} \div 3) \times 30 = 385$.

2. *Answer:* 3696 sq. ft. Diameter $= 42$ ft. $42 \times 3\frac{1}{7} = 132$, circumference of cone at base. $(132 \div 2) \times 35 = 2310$

sq. ft., which is area of curved surface. Radius of base = 21 ft. $21 \times 21 \times 3\frac{1}{7} = 1386$, area of base.

3. *Answer:* 4070 sq. in.
4. *Answer:* 21 in.
5. *Answer:* 7 in.
6. *Answer:* \$ 20.38.
7. *Answer:* $61\frac{1}{9}$ sq. yd.
8. *Answer:* 1232 cu. in.
9. *Answer:* $148\frac{1}{2}$ lb.
10. *Answer:* 12,936 cu. in.

Lesson No. 14

1. *Answer:* 154 sq. in. $49 \times 3\frac{1}{7}$.
2. *Answer:* $179\frac{2}{3}$ cu. in. $(\frac{1}{3}$ of $154) \times 3\frac{1}{2}$.
3. *Answer:* 2464 sq. ft. Diameter squared multiplied by $3\frac{1}{7}$.
4. *Answer:* 264 sq. ft. The two hemispheres forming the ends of the cylinder make one sphere of 6 ft. in diameter; the area is $113\frac{1}{7}$ sq. ft. The area of the cylinder itself is $150\frac{6}{7}$ sq. ft.
5. *Answer:* $3\frac{1}{2}$ in.
6. *Answer:* 22.4583 cu. ft.
7. *Answer:* $583\frac{1}{12}$ lb.
8. *Answer:* 840 bullets.
9. *Answer:* 5 spheres.
10. *Answer:* 368 lb.
11. *Answer:* 3 in.
12. *Answer:* $1437\frac{1}{3}$ cu. in.

Lesson No. 15

1. *Answer:* 36 in., 24 in., 18 in.
2. *Answer:* 25 times.
3. *Answer:* 1500 lb.
4. *Answer:* 54 lb.
5. *Answer:* 64 times; 512 times.

III

EASY LESSONS IN GEOMETRICAL DRAWING

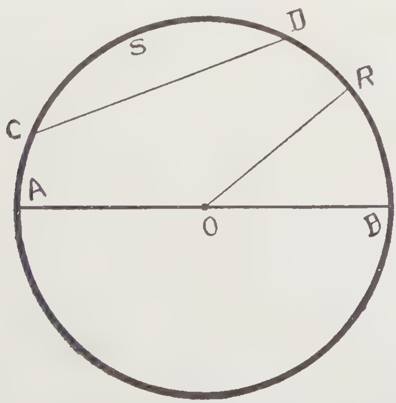
EASY LESSONS IN GEOMETRICAL DRAWING

Lesson No. 1

Students should provide themselves with the following **drawing instruments**:

1. A good ruler.
2. A ruling pen.
3. An ink compass for making circles.
4. A protractor for measuring angles.

The boundary line of a circle is called its **circumference.** The circumference is divided into 360 equal parts, called



degrees. Each degree is divided into 60 equal parts, called **minutes**; and each minute into 60 equal parts, called **seconds.**

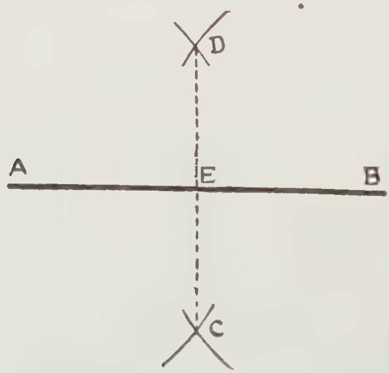
One-fourth of a circle is called a **quadrant**, or **right angle**, and is written 90° (ninety degrees). An angle thus measured would be expressed in degrees, minutes, and seconds, and written, for instance, as $37° 25' 14''$.

Every point in the circumference of a circle is equally distant from a point within called the **center**.

The **radius** is the distance from the center to any point in the circumference, as OR. The **diameter** is any line drawn through the center and terminating in the circumference, as AB. An **arc** is any portion of the circumference, as CSD. A **chord** is a line joining any two points in the circumference, as CD.

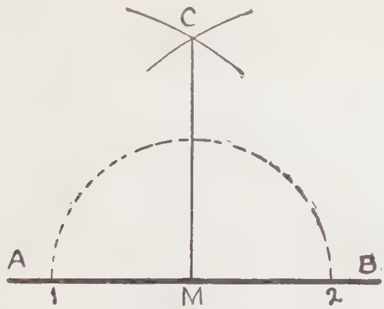
PROBLEMS

NOTE. — The figures are given for each of the problems. The student is expected to follow the instructions and draw these figures for himself.



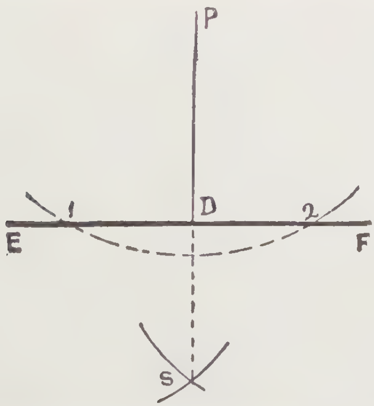
1. *To bisect a given straight line.*

Given line AB. With A and B as centers, and with any convenient radius greater than one-half of AB, describe arcs intersecting on both sides of AB at C and D. Join the points C and D, and the intersection of this line with AB will give the required point E.



2. *To draw a perpendicular line to a given line at a given point in the line.*

Given line AB, and let it be required to draw a perpendicular to AB at the point M. From M as a center, with any convenient radius, describe arcs intersecting AB at 1



and 2. Then with 1 and 2 as centers, with any radius greater than the distance from 1 to 2, describe arcs intersecting at C. Then CM will be the required perpendicular.

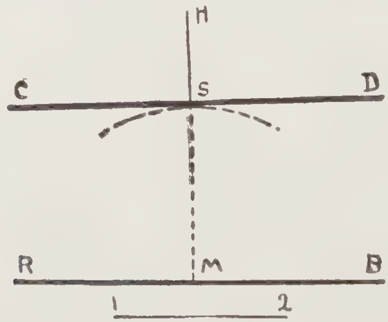
3. *To draw a perpendicular to a given line from a point without the line.*

Given line EF, and P any point without the line. From P as center, with radius sufficiently long, describe an arc intersecting EF at 1 and 2. Then with 1 and 2 as centers, and with any radius greater than the distance from 1 to 2, describe arcs intersecting at S. Join P and S intersecting EF at D. Then PD is the required line.

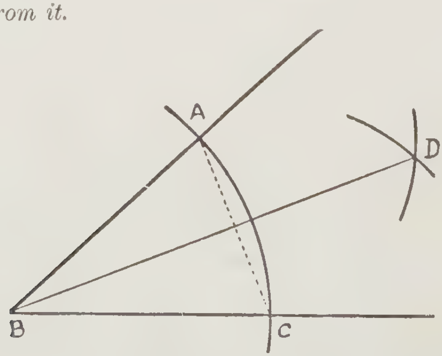
Lesson No. 2

PROBLEMS

NOTE.—For convenience of reference the numbering of the problems is continuous throughout the course.



4. *To draw a line parallel to a given line and at a given distance from it.*

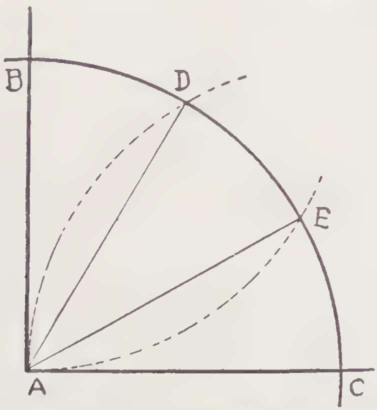


Given RB, and as distance the line drawn from 1 to 2. At any point M in RB draw a perpendicular MH, as shown

in Problem 2. Lay off on MH with radius equal to the line from 1 to 2 the distance MS. At S in MH draw CD perpendicular to MH; then CD will be the required line.

5. *To bisect a given angle.*

Let B be the given angle. From B as a center, with any convenient radius, describe the arc AC. From A and C as centers, with convenient radius, describe arcs inter-



secting at D. Join B and D. Then BD is the line bisecting the angle B. BD bisects the chord AC and also the arc. The arc measures the angle, therefore the angle is bisected.

6. *To trisect a right angle.*

Given the right angle A. From A describe any convenient arc BC. Then with the same radius describe arcs, with B and C as centers, intersecting arc BC at D and E respectively. Join D and E with A. Then angles BAD, DAE, and EAC are equal, and the right angle A is trisected.

Lesson No. 3

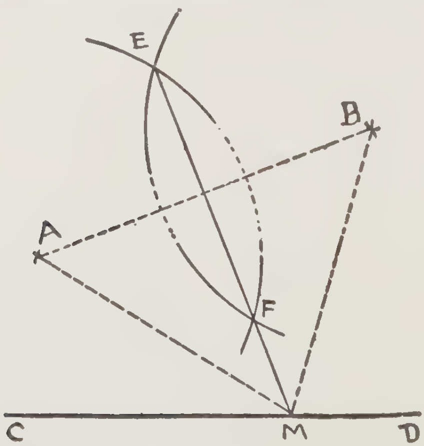
See to it that your drawing instruments are kept clean and in good order. Important drawings should be first made in pencil and inked afterward. In this way the lines

can be made of accurate lengths and distances. Make the pencil lines very light, so that errors can easily be rubbed out. In using the compass the lower part of the legs should be kept nearly vertical, so that the needle point will make only a small hole in revolving, and both nibs of the pen may press equally on the paper. Students will find it to their advantage to have a draughtsman's T-square, and also a right triangle for ruling and measuring.

PROBLEMS

7. *To find a point in a straight line equally distant from two given points without the line.*

Given a line CD and two fixed points A and B without the line. With A and B as centers, and with radius sufficiently large, describe arcs intersecting at E and F. Join

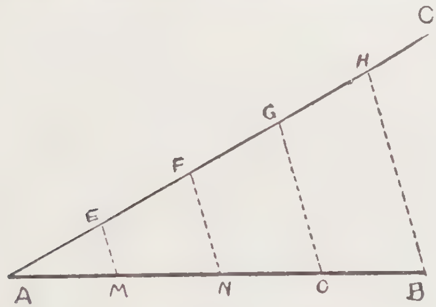


E and F and produce the line to meet CD in M. Then M is the required point.

Every point in the perpendicular bisector of a line is equally distant from its extremities. M is such a point with reference to the line AB. Therefore M is equally distant from the points A and B.

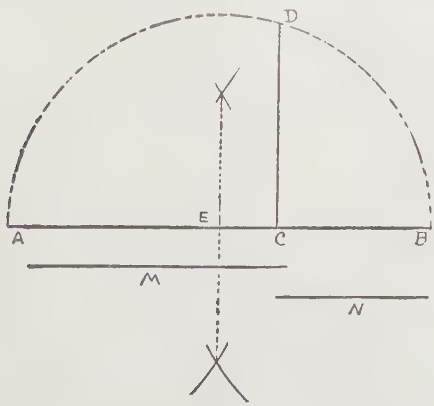
8. *To divide a line into any number of equal parts.*

Given any line *AB*, and let it be required to divide it into four equal parts. Draw any line, as *AC*, making any convenient angle with *AB*. With any convenient measure of length lay off four equal parts on *AC*, as *AE*, *EF*, *FG*,



GH. Join *H* with *B*, and then draw lines from *G*, *F*, and *E* parallel to *HB*. Then *M*, *N*, and *O*, the points of division, will divide *AB* into four equal parts. In a similar manner *AB* could be divided into five or any other number of equal parts.

9. *To find a mean proportional between two given lines.*



A mean proportional between M and N is a line such that in respect to length it has the same relation to N that M has to *it*.

Upon an indefinite line AB lay off AC equal to M, and CB equal to N. Bisect AB in E, and with center E and radius EB draw a semicircle. At C draw CD perpendicular to AB. Then CD will be the mean proportional required.

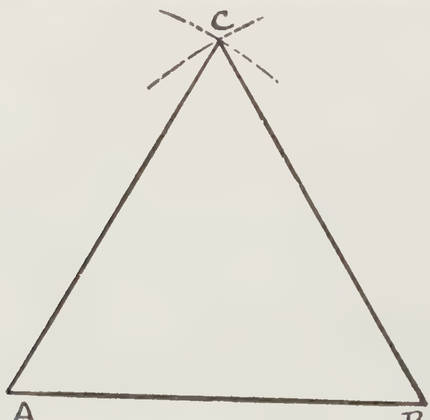
Lesson No. 4

A **triangle** is a plane figure bounded by three straight lines. When the sides are all equal the triangle is said to be **equilateral.** When two sides only are equal the triangle is said to be **isosceles.**

PROBLEMS

10. *To construct an equilateral triangle on a given straight line.*

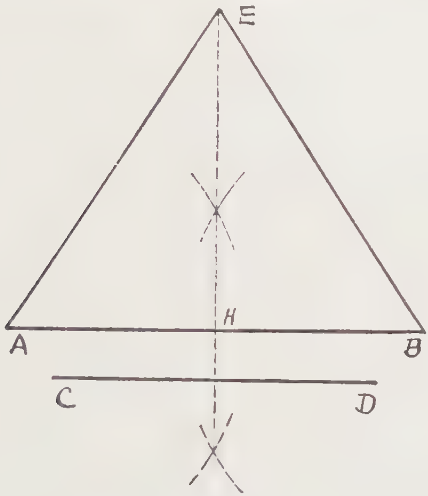
Let AB be the given straight line. With A and B as centers, and the line AB as radius, describe arcs intersecting



at C. Join C with A and B. Then ABC is the triangle required.

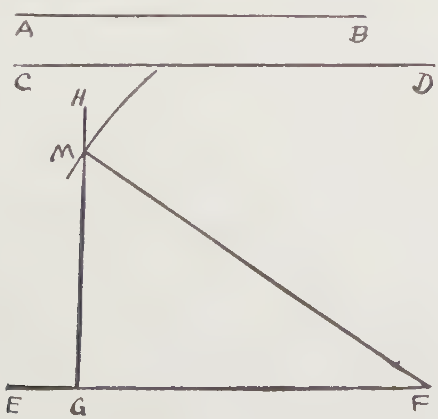
11. *To construct an isosceles triangle, the base and altitude being given.*

Let AB be the given base, and CD the altitude. Bisect the base AB with the perpendicular EH (see Problems 1



and 2), and make EH equal to CD. Join EA and EB. Then EAB is the required triangle.

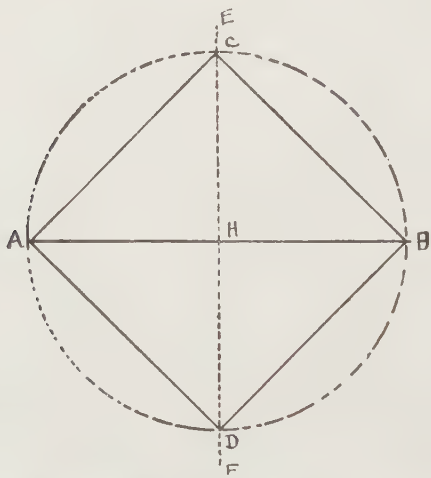
12. *To construct a right-angled triangle having given the hypotenuse and base.*



Let AB be the base, and CD the hypotenuse. Draw any straight line EF. Lay off GF equal to AB. At G erect a perpendicular GH. Then with F as center, and with radius equal to CD, describe an arc intersecting HG at M. Draw MF. Then MFG is the required triangle.

13. *To construct a square, the diagonal being given.*

Let AB be the diagonal. Bisect AB by the perpendicular EF. With H as center, and radius HB, describe arcs



cutting the perpendicular in C and D. Join AC, CB, BD, and AD. Then $ACBD$ is the required square.

Lesson No. 5

PROBLEMS

14. *To find a fourth proportional to three given lines.*

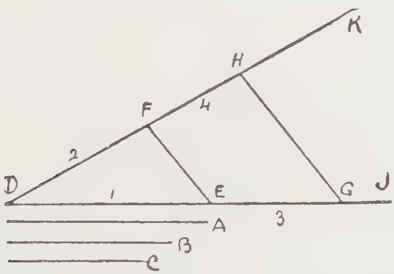
Let A, B, and C be the three given lines. Draw DJ and DK, of indefinite length and at any angle to each other. Set off DE equal to A, DF equal to B, and EG equal to C.

Join EF. From G rule a parallel to EF, cutting off FH. FH is the fourth proportional required.

$$DE : DF :: EG : FH.$$

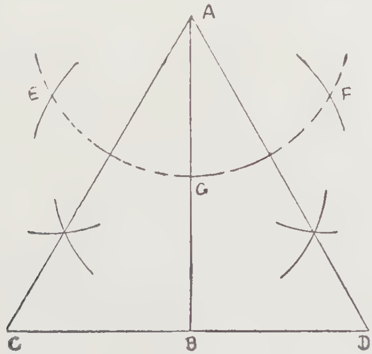
That is, $$A : B :: C : FH.$$

This construction answers for all cases. It should be worked to a scale and the result verified by the student. If



the three lines be 3, $2\frac{1}{2}$, and 2 in. long, then the fourth proportional will be as in $3 : 2\frac{1}{2} :: 2 : FH$. This relation gives us $3\,(FH) = 2\frac{1}{2} \times 2$. Therefore, $FH = (2\frac{1}{2} \times 2) \div 3 = 1\frac{2}{3}$.

15. *To construct an equilateral triangle, the altitude being given.*

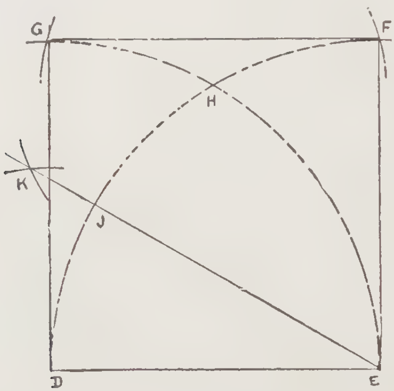


Let AB be the given altitude. At B draw CD, of indefinite length, at right angles to AB. With center A, and

any radius, describe an arc cutting AB in G. With center G, and radius equal to GA, describe arcs cutting the arc EGF in E and F. Bisect EG and FG by the lines AC and AD, meeting CD in C and D. Then ACD is the required triangle.

NOTE. — The student should prove that the angles at C and D are each 60° and the angles at A each 30°.

16. *To construct a square from a given side.*



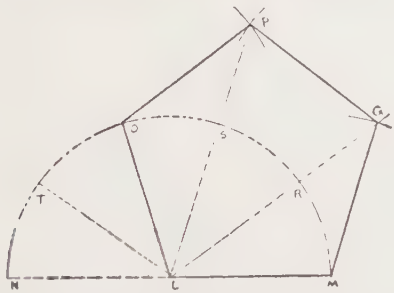
Let DE be the given side. With one foot of the compasses on E, and ED as radius, describe the arc DF. Similarly, with D as center, describe the arc EG, cutting the arc DF at H. Bisect DH at J by the line EK. Make HF and HG each equal to HJ. Connect EF, FG, and GD to complete the square.

NOTE. — The student should prove that the angles DEJ, JEH, and HEF are each 30°; also that the angle HDE is 60°.

17. *To construct a pentagon from a given side.*

Let LM be the given side. With L as center describe the semicircle MON. Divide MON into five equal parts, and from the center L draw radii through the divisions T, O, S, and R, as shown in the diagram. With LM as radius, and M and O as centers, describe arcs cutting LSP and LRQ at

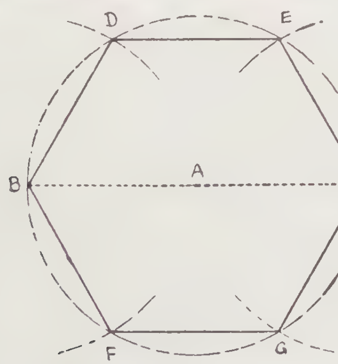
P and *Q*. Draw the lines *OP*, *PQ*, and *QM* to complete the pentagon.



NOTE. — The division of the semicircle *MON* into five equal parts has not in these lessons been shown how to be done. In practice it will generally be found easy to do by trial.

18. *To construct a hexagon from a given side.*

With *A* as center, and radius *AC* equal to the given side, describe a circle. Draw the diameter *BC*. With radius



AC, and centers *C* and *B*, describe arcs intersecting the circle at *D*, *E*, *F*, and *G*. Join *BF*, *FG*, *GC*, *CE*, *ED*, and *DB*, and the hexagon is complete.

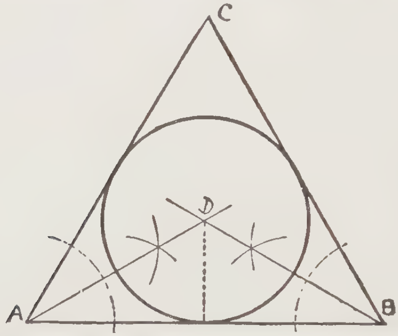
NOTE. — The student should prove that the angles at *A*, subtended by the sides of the hexagon, are each the angles of an equilateral triangle, and that the hexagon divides into six equilateral triangles.

Lesson No. 6

PROBLEMS

19. *To inscribe a circle in a given triangle.*

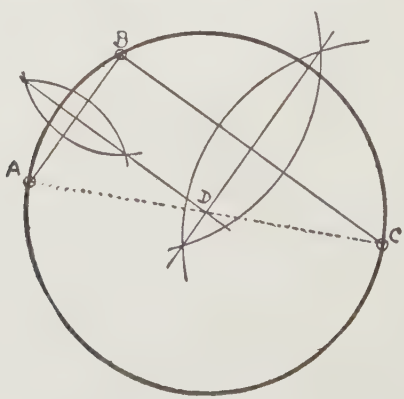
Let ABC be the triangle. Bisect any two of the angles. Produce the bisecting lines to meet in D. From D draw DE perpendicular to the side AB. Then from D, with



radius DE, a circle may be described which will touch all three sides of the triangle. This is called the inscribed circle.

20. *To draw a circle through three points.*

The points must not be in the same straight line. Let A, B, and C be the points. Join AB and BC. Then bisect

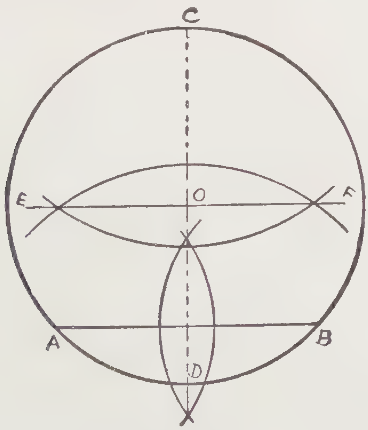


AB and *BC*, and produce the bisecting lines until they cut each other in the point *D*. Then *D* will be equally distant from each of the three points. Therefore from *D*, with radius *DA*, *DB*, or *DC*, a circle may be drawn which will pass through the three given points.

If *AC* is joined, the figure *ABC* is a triangle, and thus the solution of this problem serves also for describing a circle which shall pass through the three angular points of a triangle. Such a circle is called a circumscribing circle.

NOTE. — In the figure above the point *D*, or center of the circle, falls upon the line *AC*. This can be so only when the angle *ABC* is a right angle.

21. *To find the center of a circle.*

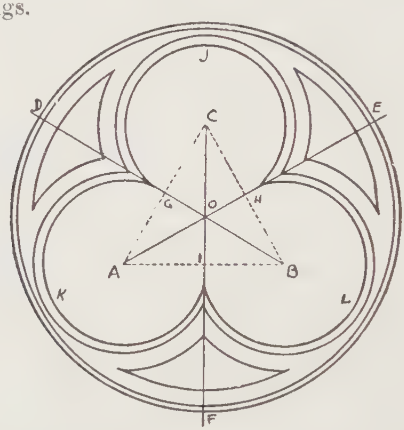


Take any circle, draw a chord as *AB*, and bisect it by a line cutting the circle in *C* and *D*. Bisect *CD* by the line *EF*. The intersection *O* is the center of the circle.

THE GOTHIC TREFOIL

The **trefoil** is a figure much used in Gothic architecture. It is formed of three leaves, or lobes, meeting at a center, as in the three-leaved clover. It is sometimes inclosed in a

circle, as in window tracery, but not always, as in many wall piercings.



This figure will serve as an application of the construction of the equilateral triangle and the bisecting of angles. It is here introduced with the view of showing students the importance of *absolute* accuracy in the working out of their problems.

Construct an equilateral triangle, ABC. Bisect the angles by the bisectors AE, BD, CF, meeting in O. Observe that in an equilateral triangle the lines which bisect the angles will bisect the sides opposite to the angles as well, and thus the points G, H, I are obtained. From A, B, and C, with radius AG, equal to half the side of the triangle, describe the arcs J, K, L, and the others, which, it will be plainly seen, are concentric with them. The outer circles are drawn with O as center.

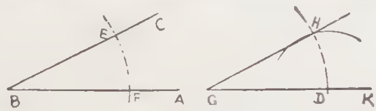
Lesson No. 7

PROBLEMS

22. *To construct upon a given line an angle equal to a given angle.*

Let CBA be the given angle and GK the given line. From B, with any radius, describe the arc EF. From G,

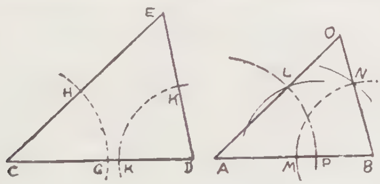
with the same radius, describe an arc cutting GK in D. With radius EF, and center D, describe an arc cutting HD



in H. Join HG. Then the angle HGD will equal the angle CBA.

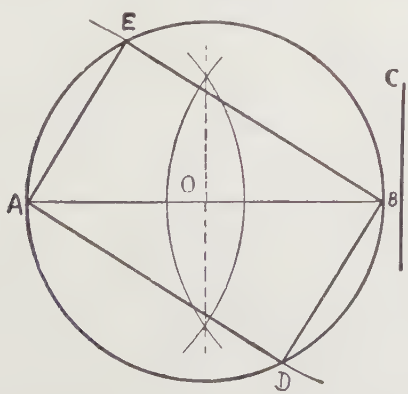
23. *On a given line to construct a triangle similar to another triangle.*

Let ECD be the given triangle and AB the given line. At A construct an angle OAB equal to ECD, and at B con-



struct an angle OBA equal to EDC. Let AO and BO meet in O. Then OAB is a triangle similar to ECD.

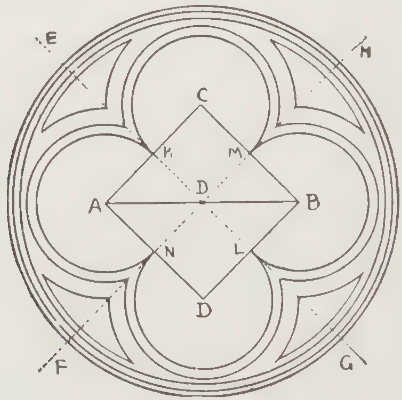
24. *To construct a rectangle where the diagonal and the length of one pair of sides are given.*



Let AB be the diagonal and C the given length. Bisect AB at O. From O, with radius OA, describe a circle. From A and B set off the length of the line C on the circle; viz. AE and BD. Join the points A, E, B, D, and the required figure will be completed.

THE GOTHIC QUATREFOIL

We learned last lesson how to construct a Gothic trefoil. The quatrefoil is formed of four leaves or lobes, meeting in the center.



Construct the square $CADB$ upon the diagonal AB. Bisect the sides by the lines EG and FH, cutting the sides at K, M, L, and N. From A, C, B, and D, with radius AK, that is, half the side of the square, draw the arcs, and with slightly larger radius draw the concentric arcs. The outer circles are drawn from the center D.

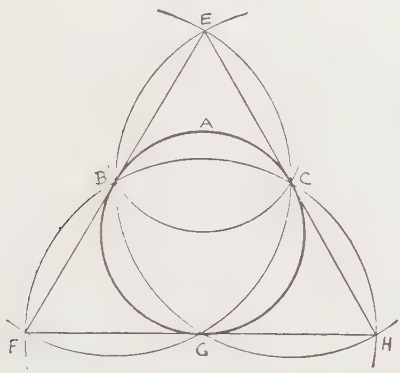
Lesson No. 8

PROBLEMS

25. *To construct an equilateral triangle about a given circle.*

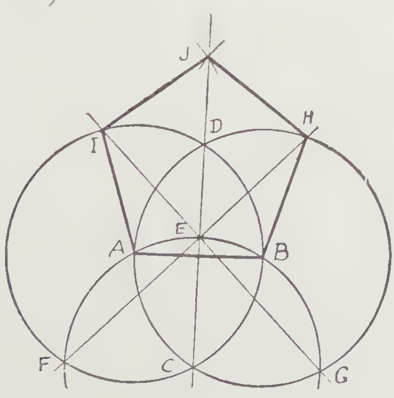
From any point in this circle, as A, with a radius equal to the radius of the circle, describe the arc cutting the circle

in B and C. From B and C, with radius BC, draw the semicircles cutting each other at E and G. With G as center and the same radius draw the semicircle $FBCH$. Then F, H, and E are corners of the required triangle.



NOTE. — The student will notice that the two circles with B and C as centers that cut each other at G cut the original circle also at G. He will also notice that the straight lines EG, FC, HB, are at right angles to FGH, HCE, and EBF, respectively.

26. *To construct a regular pentagon on a given line.* (Special method.)



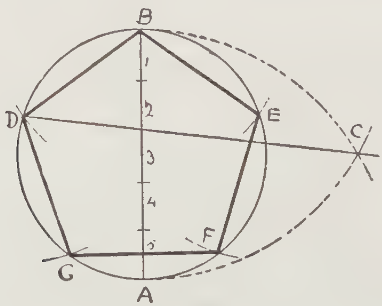
Let AB be the given line. From A and B, with radius AB, describe arcs cutting each other in C and D. Draw a line through C and D. From C, with radius CA (equal to AB), describe an arc cutting the perpendicular in E, and also cutting the two previously drawn circles in F and G. From F draw the line FEH, and from G the line GEI. Draw AI and BH. From H and I, with radius equal to the side of the pentagon, describe arcs cutting each other in J. Draw JH and IJ, which will complete the pentagon.

Lesson No. 9

PROBLEMS

27. *To inscribe a regular polygon — in this case a pentagon — in a given circle.*

Draw the diameter AB and divide it into as many equal parts as the polygon is to have sides (in this case five). From A and B, with radius AB, describe arcs cutting each other in C. From C draw a line passing through the *second* division and cutting the circle in D. Draw DB, which will



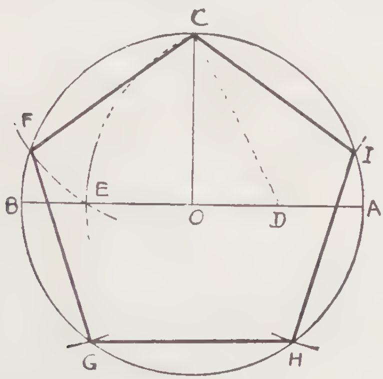
be one side of the polygon. Set off the length DB around the circle; viz. E, F, G. Join these points and thus complete the figure.

Any polygon may be thus formed, by dividing the diameter into as many parts as the polygon has sides, but the

line CD must, in every case, be drawn through the second division.

28. *To inscribe a regular pentagon in a circle.* (Special method.)

Draw the diameter AB. At O erect a perpendicular OC. Bisect OA in the point D. From D, with radius DC, describe an arc cutting AB in E. From C, with radius CE,

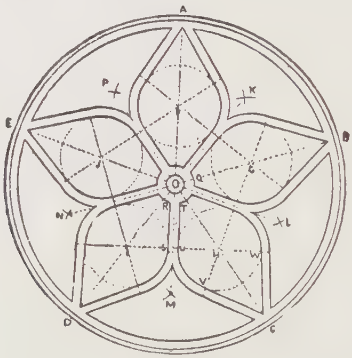


describe an arc cutting the circle in F. Draw CF, which will be one side of the pentagon. Set off the length CF around the circle; viz. G, H, I. Draw FG, GH, HI, and IC, which will complete the figure.

APPLICATION OF FOREGOING PRINCIPLES IN THE CONSTRUCTION OF GOTHIC TRACERY

Draw a circle; divide it into five equal parts. Draw the five radii. Bisect one of the radii and set off the half on each. Join the five points marking the half-radii, forming a regular pentagon. Bisect the sides of this pentagon. Draw a small circle in the center and another concentric with it. From the outer of these small circles draw parallel

lines and complete the figure by drawing the remaining lines as shown in the diagram.

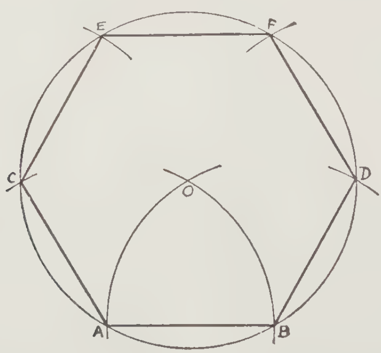


Lesson No. 10

PROBLEMS

29. *To construct a regular hexagon on a given line.* (Second method. See Problem 18.)

Let AB be the given line. From A and B, with radius AB, describe arcs cutting each other in O. From O, with

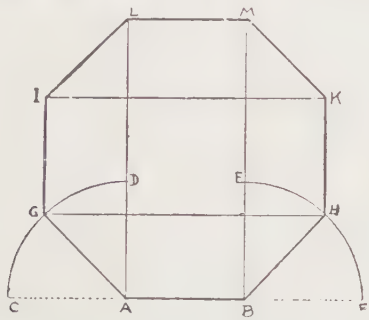


radius OA or OB, describe a circle. The radius with which a circle is struck will divide it into six equal parts;

therefore, set off the lengths AC, CE, EF, FD, and DB. Join these points, and a hexagon will be formed.

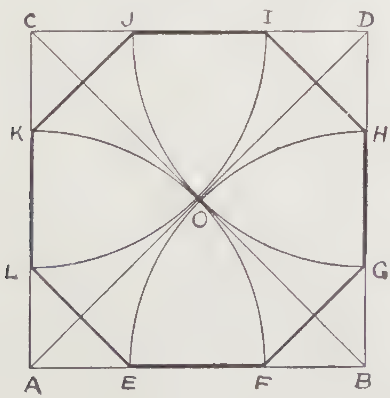
30. *To construct a regular octagon on a given line.*

Let AB be the given line. Produce AB to C and F. Erect perpendiculars at A and B. From A and B, with radius AB, describe the quadrants CD and FE. Bisect these quadrants; then AG and BH will be two more sides



of the octagon. Join G and H. At H and G erect perpendiculars GI and HK, equal to AB. Draw IK. Make the perpendiculars AL and BM equal to GH or IK. Draw IL, LM, and MK. These will complete the octagon.

31. *To inscribe an octagon in a square.*



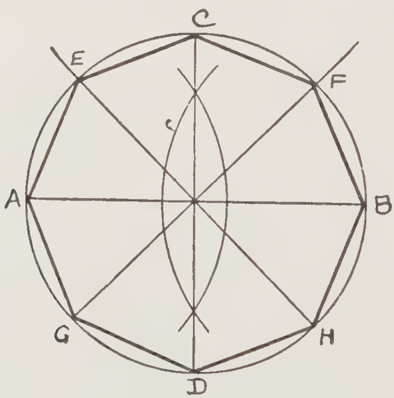
Let $ABCD$ be the squares. Draw diagonals intersecting at O. From A, B, C, and D, with radius AO, or BO, or CO, or DO, describe four quadrants cutting the sides of the square. Join the points thus found, and the octagon will be complete.

Lesson No. 11

PROBLEMS

32. *To inscribe an octagon in a given circle.*

Draw the diameter AB and bisect it at right angles by CD. Bisect the quadrants AC, CB, AD, and BD in the points E, F, G, H, by straight lines passing through the

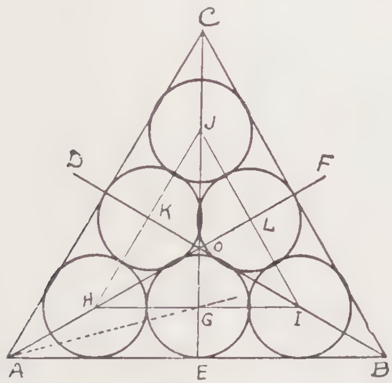


center of the circle. Draw lines connecting all the eight points where these bisectors meet the circle. These will complete the required octagon.

33. *Within an equilateral triangle to inscribe six equal circles.*

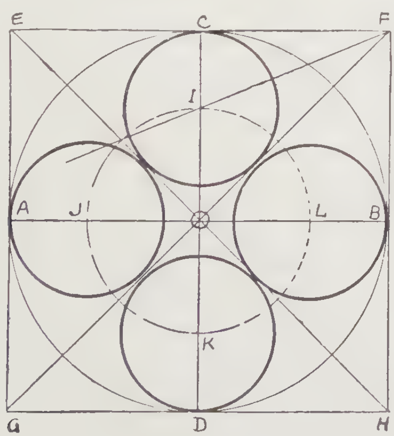
Let ABC be the triangle. Draw the lines BD, AF, and CE, bisecting the sides and angles of the triangle and intersecting each other in O. Bisect the angle OAE. The point (G) where the bisecting line cuts CE will be the

center of one of the three isosceles triangles into which the equilateral triangle has been divided. Through *G* draw *HI*, parallel to *AB*, and from *H* and *I* draw *HJ* and *IJ*



parallel to *AC* and *BC* respectively, cutting *BD* and *AF* in *K* and *L*. With *H*, *G*, *I*, *J*, *K*, and *L* as centers, and radius *GE*, draw the six circles.

34. *To inscribe four equal circles in a circle, each touching two others and the containing circle.*



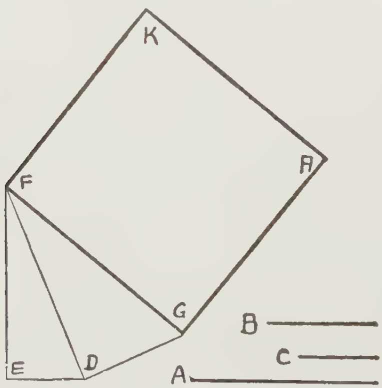
Let *ADBC* be the circle. Draw the diameters *AB* and *CD* at right angles to each other. From *A*, *B*, *C*, and *D*, with radius *AO*, describe arcs cutting each other in *E*, *F*, *G*, and *H*. Join these points, and a square will be described about the circle. Draw the diagonals *EH* and *GF*. Bisect the angle *CFO*, and produce the bisecting line until it cuts *CD* in *I*. From *O*, with radius *OI*, describe a circle cutting the lines *AB* and *CD* in *J*, *K*, *I*, and *L*. From these centers, with radius *IC*, describe the four required circles.

Lesson No. 12

PROBLEMS

35. *To construct a square equal in area to the sum of three squares.*

Let *A*, *B*, and *C* be the lengths of the sides of the given squares. Draw *EF* equal to *A*, and *ED* at right angles to *EF*, and equal to *C*. Join *FD*. Then the square on *FD* is



equal to the sum of the squares on *FE* and *ED*. This is proven in geometry. Draw *DG* at right angles to *DF*, and equal to *B*. Join *FG*. On *GF* describe a square. Then,

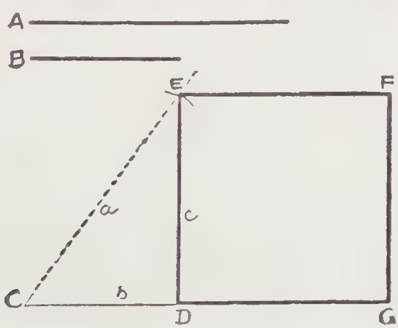
$$(FG)^2 = (FD)^2 + (DG)^2 = (FE)^2 + (ED)^2 + (DG)^2$$

= the square on A + the square on C + the square on B.

Therefore, $FGHK$ is the required square.

36. *To construct a square equal in area to the difference between two given squares.*

Let A and B be the sides of the given squares. Draw two lines, DC and DE, at right angles to each other. Make DC equal to B, and, with C as center and radius equal to A,



mark off CE, making the right-angled triangle EDC. Then the square on ED, namely, $EDGF$, will be the required square. This is shown by algebra very clearly as follows:

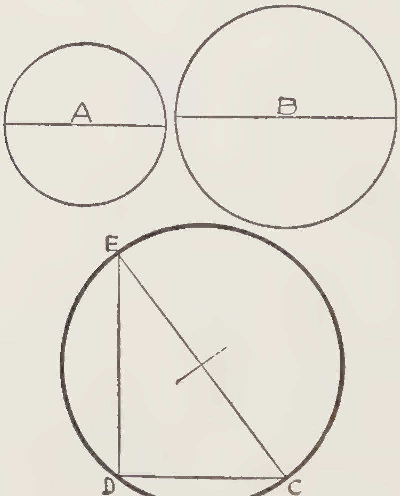
$$a^2 = b^2 + c^2;$$

therefore, $$c^2 = a^2 - b^2.$$

37. *To describe a circle equal in area to the sum of two given circles.*

Let A and B be the two given circles. Draw CD and DE perpendicular to each other, and equal to A and B, the diameters of the given circles. Draw EC. Then EC is the diameter of the required circle.

The proof of this is established thus: By geometry we know that the square on EC is equal to the sum of the squares on ED and DC. But similar figures are to one another as the squares of their similar lines. Therefore, the



area of the circle that has EC as its diameter is equal to the sum of areas of the circles that have ED and DC as their diameters.

To describe a circle equal in area to the difference between two circles, proceed as in Problem 36.

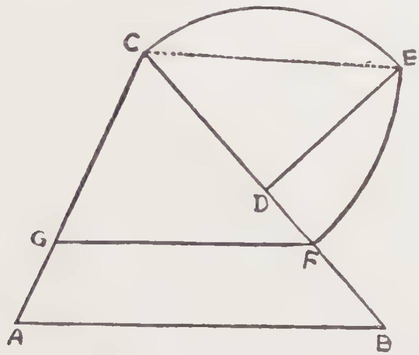
Lesson No. 13

PROBLEMS

38. *To divide a given triangle into two equal parts by a line parallel to one of its sides.*

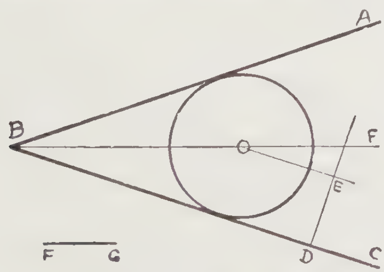
Let ABC be the triangle. Bisect one of the sides, as CB, in the point D, and erect the perpendicular DE equal to DC. From C, with radius CE, describe an arc cutting CD

in F. From F, draw FG parallel to AB. FG will divide the triangle into two parts of equal area.



39. *To draw a circle of a given radius which shall touch both lines of an angle.*

Let ABC be the angle, and FG the given radius. Bisect the angle by the line BF. On either line of the angle, say BC, erect a perpendicular DE equal to FG. From E draw

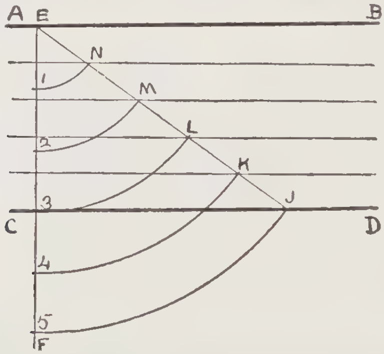


a line parallel to BC, cutting the bisecting line in O. From O, with the given radius, draw the circle, which will touch both the lines of the angle.

40. *To divide the space contained between two parallel lines into a given number of equal parts.*

Draw the line EF perpendicular to AB, and set off on it equal lengths corresponding to the number of spaces into

which $ABCD$ is to be divided, in this instance, *five*. These spaces may be any length, but must be equal. From E, with radius $E\,5$, describe an arc cutting CD in J. Draw



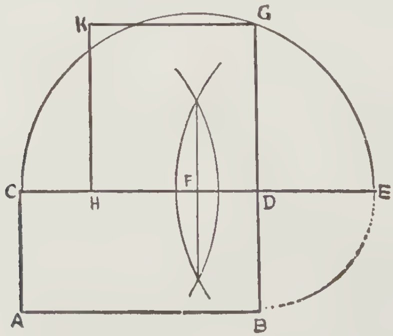
EJ. From E, with radius $E\,4$, $E\,3$, $E\,2$, $E\,1$, describe arcs cutting EJ in K, L, M, N. Draw lines parallel to AB through these points, and the space will be divided as required.

Lesson No. 14

PROBLEMS

41. *To construct a square equal in area to a rectangle.*

Let $CABD$ be the rectangle. Produce CD until DE equals DB. Erect a perpendicular at D. Bisect CE in F.

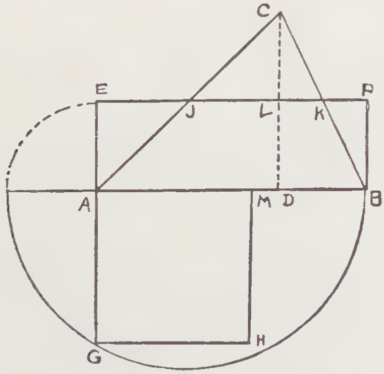


From F, with radius FC, describe a semicircle cutting the perpendicular erected at D in G. DG will be the length of the side of the required square. From D set off DH equal to DG. From H and G, with radius DH, describe arcs cutting each other in K. Draw HK and GK. These will complete the square.

NOTE. — This is but a particular use of the general proposition that when two chords intersect one another in a circle, the rectangle contained by the two parts of the one is equal to the rectangle contained by the two parts of the other.

42. *To construct a rectangle equal in area to a given triangle.*

Let ABC be the given triangle. From C drop a perpendicular CD. Erect perpendiculars at A and B. Bisect CD



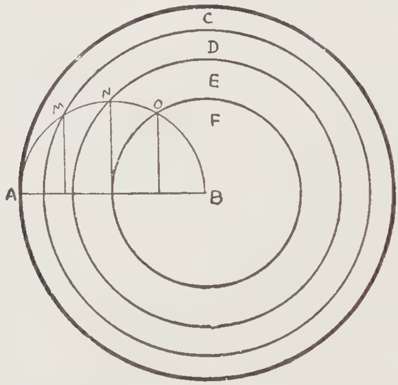
in L. The bisecting lines cutting the perpendiculars at E and F, will complete the rectangle equal in area to the triangle.

NOTE. — It is plain that the rectangle $EABF$ is half of the rectangle whose base is AB and altitude CD; also that the triangle CAB is half of the same rectangle.

A square $AGHM$, equal in area to the rectangle $EABF$, constructed as per last figure, will, of course, be equal to the triangle.

43. *To divide a circle into any number of concentric circles having a constant difference of area.*

Draw a radius AB, and on it describe a semicircle. Divide the radius AB into the number of equal parts corresponding with the number of circles required. From the points of



division raise perpendiculars cutting the semicircle in M, N, O. Through these points, from B as a center, draw circles. Then the belts C, D, and E and the circle F will have the same area.

NOTE.—The student may prove this if he wishes by the application of three principles: (1) That the areas of circles are to one another as the squares of their radii; (2) that in right-angled triangles the square on the hypotenuse is equal to the sum of the squares on the sides; and (3) the principle contained in the Note to Proposition 41 above. Try it when AB is divided into five equal parts.

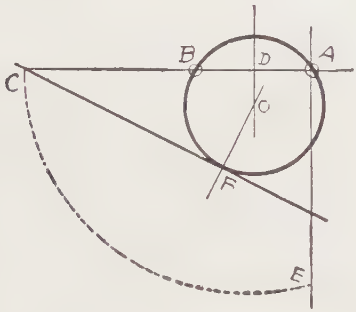
Lesson No. 15

PROBLEMS

44. *To describe a circle which shall pass through two given points and touch a given straight line.*

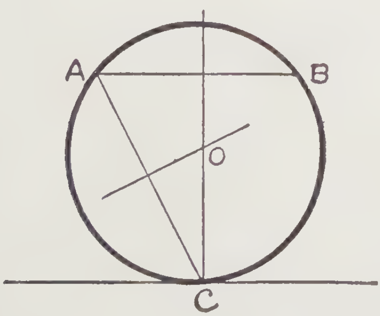
Join the points A and B and produce the line to meet the given line in the point C. Find a mean proportional between

AC and CB by bisecting AB, describing an arc with D as center and DC as radius and drawing the perpendicular AE. Make CF equal to AE. At F draw the perpendicular FO



to meet the perpendicular from D. With center O and radius OF describe the circle.

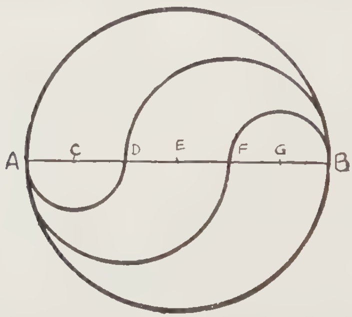
In the case where the line joining the two points is parallel to the given line, join the two given points A and B.



Bisect AB by a perpendicular line meeting the given straight line in C. Draw AC and bisect it by a perpendicular line. With center O, and radius OC, describe the circle.

45. *To divide a circle into any number of parts equal in area and perimeter.*

Divide the diameter AB of the circle into twice as many parts as are required. If the circle is to be divided into three equal parts, divide AB into six equal parts, as shown in the figure. With centers C and G, and radius one-



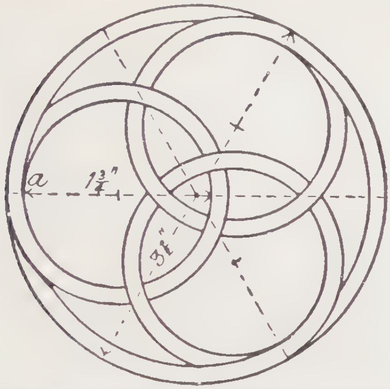
sixth of AB, describe the semicircles AD and FB; with centers F and D, and radius one-third of AB, describe the semicircles AF and DB. The circle will then be divided into three parts of equal area and perimeter.

Lesson No. 16

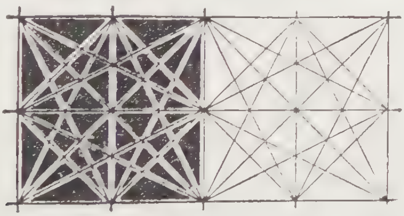
CONCLUDING NOTES

The subject of geometry and geometrical drawing are of the largest value in all kinds of constructive work. It is difficult to name the technical study in which the principles of geometry are not exemplified; its lines, forms, or figures are met with everywhere and its problems involve principles upon which some of the most important work of man is dependent. Notice the following very simple illustrations:

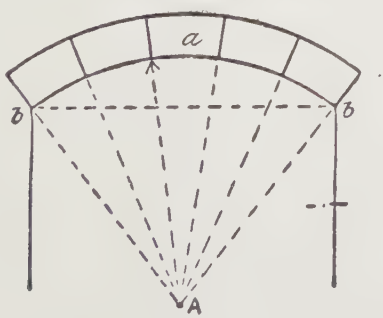
Here we have a geometrical pattern for a window, drawn in accordance with the principles already set forth in these lessons.

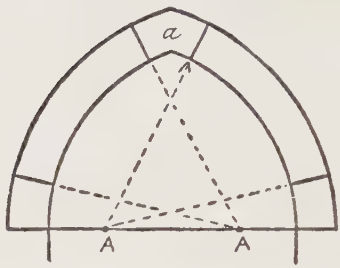


This figure shows a design made wholly of straight lines. An infinite number of beautiful patterns may be created in this way by introducing other lines and varying their direction.

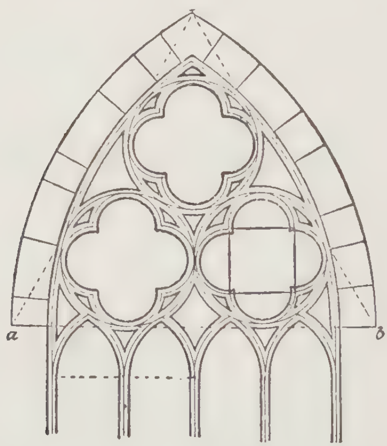


These two examples of window arches are constructed upon simple geometrical principles.





The window design shown in this illustration is constructed by first describing the equilateral triangle ABC; in this triangle are drawn the three largest possible circles.



Then from the centers A and B the outer lines are described; within each inner circle are inscribed four semicircles, and then the tracery filled in as shown.

It is clearly seen that many of the familiar and varied forms which are the product of manual skill or of machinery have in them in original design the principles of geometry. The bricklayer or stonemason cannot erect a wall, or set out the directions which it assumes in relation to other walls,

without availing himself of methods which are essentially geometrical in their principles. The same is true of carpenter, cabinet-maker, machinist, and engineer. The metal-worker, the coppersmith, the tinsmith, the boiler-maker, cannot cut out on the flat the pieces of material which, when bent or curved or placed together, assume the forms which they desire to make, without availing themselves of the principles of geometry.

Workshop methods have been handed down from one generation to another, and the majority of workmen do their work in complete ignorance of the fact that the figures they construct are based upon the principles of mathematics. Hence it follows that when work involving new designs is to be constructed they are at a loss to know how to proceed. If to their manual dexterity and practiced training workmen could add a knowledge of the principles upon which their work is based, the work would be all the better, because more intelligently done; and with increase of intelligence as to the nature of work will come an increase of interest in doing it, and the doing of it better will be the natural result.

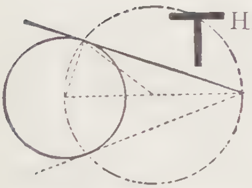
It is the union of practice with theory which gives to practice its highest value. Educated labor is always in demand.

IV

ELEMENTARY ALGEBRA

ELEMENTARY ALGEBRA

Lesson No. 1. Introductory



THE elementary processes of **algebra** differ but slightly from those of arithmetic. In arithmetic numbers are represented by figures, the values of which are known. In algebra numbers are represented largely by letters which have an unknown or an unassigned value. The following arithmetical symbols, with some others which will be introduced later, are used in algebra:

$+$ **plus**, the sign of addition.

$-$ **minus**, the sign of subtraction.

$\times$ multiplied by.

$\div$ divided by.

$\sqrt{}$ square root of.

$=$ is equal to.

When two figures are written alongside of each other, the first figure represents **tens**, and the second **units**. In algebra, when two or more letters are written alongside of each other, it indicates that the numbers represented by the letters are **to be multiplied together**. Thus:

$$47 \quad = 40 + 7.$$

$$ab \quad = a \times b.$$

$$347 = 300 + 40 + 7.$$

$$3\,ab = 3 \times a \times b.$$

In the last illustration the 3 is called a **coefficient**. When the coefficient is 1, it is omitted. If we want to indicate $a + a + a$ in algebra, we write $3a$, which means 3 times a. Thus in $3abc$ the 3 denotes that the product of a, b, and c is to be taken 3 times.

An **algebraic expression** is a quantity expressed in algebraic language.

Thus $$2ab + 3xy - 6m + 5$$
is an algebraical expression. This expression is read as follows:

Two ab, plus three xy, minus six m, plus five.

The expression $$6a(a+b) - cd$$
is read

Six a, into quantity a plus b, minus cd.

The expression $$2a^2 + ab - 3b^3$$
is read

Two a square, plus ab, minus three b cube.

In the second illustration the $6a$ is to be multiplied by the sum of a and b. The brackets indicate that $a + b$ represents one quantity, and this quantity is to be multiplied by $6a$. In the third illustration the $2a^2$ (read "two a square") is equal to $2 \times a \times a$. Instead of writing a twice, or three times, or four times, we write a small number called an **exponent** a little to the right and above the number. $3b^3$ (read "three b cube") is equal $3 \times b \times b \times b$.

EXERCISES

When $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, find the numerical value of the following algebraic expressions:

1. $4a + 3b + 2c - d$.
2. $2ab + 3ed + 5abcd$.
3. $2b^2 + 3ac + c^2 - e^2$.
4. $10a^3b^2 + 12c^2d + 3ab$.
5. $b^2 + c^2 + 3ab + 2a - 4$.
6. $6a(b + c + d) - 3a$.
7. $5(b + c) - 2a(d^2 - e)$.
8. $3abc - c^2 + b(3b + 4)$.
9. $(a + b)^2 + 3b(15 - c)$.
10. $(a^2 + b^2)b + 2abc$.

Lesson No. 2. Algebraic Addition

An algebraic expression whose parts are not separated by + or − is called a **term**. For example, in each of the following expressions there are three terms:

$$3x^2y - 2ab + 6c^2,$$
$$a(a+b) - cd + a^2,$$
$$5a^2 - (a+b) + \frac{a}{b}.$$

NOTE.—In the second of the above expressions, though the expression consists of three terms, the first term is in part made up of two terms. Similarly, in the third of the expressions, the second term is itself made up of two terms.

A term which has the sign **plus** before it is called a **positive** term. When the first term of an expression is positive, the sign **plus** is usually omitted. A term which has the sign **minus** before it is called a **negative** term. The sign is part of the term, and must be moved about with it if the position of the term in the expression is changed. Terms which contain the same letters and exponents are called **similar** terms. The coefficients and signs may be different, and yet the terms be similar. In the following expressions the terms are similar in each expression:

$$3ab + 4ab - ab + 6ab,$$
$$2a^2bc - 12a^2bc + 3a^2bc.$$

As in arithmetic we can combine—that is, add or subtract—only like quantities, so in algebra. Thus:

$$5 \text{ ft.} + 7 \text{ ft.} = 12 \text{ ft.},$$
$$5ab + 7ab = 12ab.$$

Five feet plus seven inches could be expressed 5 ft. + 7 in.; so in algebra $5a$ *plus* $7b$ is expressed $5a + 7b$. Note the following:

$12 - 15$, impossible in arithmetic;
$12 - 15 = -3$, in algebra.

In algebra we have enlarged our ideas of numbers to include -1, -2, -3, -4, etc., as well as 1, 2, 3, 4, etc. In other words, we count in both directions. In the illustration, if 12 represented 12 lb. pulling in a *plus* direction, and 15 represented 15 lb. pulling in a *minus* direction, the difference would be 3 in the minus direction, or -3. Thus:

$$5b - 7b = -2b,$$
$$-8a + 3a = -5a,$$
$$-6b - 3b = -9b.$$

To **add similar terms** you find the difference between the sum of the positive terms and the sum of the negative terms, giving the result the sign of the larger sum.

EXERCISES

Write the following expressions in their simplest form:

1. $8a + 7a + 12a + 3a + 16a.$
2. $-2a - 3a - 7a - 9a - 15a.$
3. $12ab - 3ab - 19ab + 20ab.$
4. $6a^2b + 9a^2 2b + 8a^2b - 30a^2b.$
5. $7bc - 6bc - 9bc - 24bc.$
6. $3m + 6m + 12m - 8m.$
7. $2(ab) + 12(ab).$
8. $2(a+b) + 3(a+b) + (a+b).$
9. $9(x-y) - 4(x-y) + 8(x-y).$
10. $a(b+c) + 2a(b+c) + 5a(b+c).$

Lesson No. 3. Simple Equations

An **equation** is a statement of equality between two expressions. Thus:

$$x + 3x + 5x = 27.$$

The parts of an equation to the right and left of the sign of equality are called the **members** or **sides of the equation,** and are distinguished as the *right side* and *left side.*

The quantities occurring in an equation are either **known** or **unknown**. The process of finding the value of the unknown quantity is called **solving the equation.** In the illustration above, x is the unknown quantity. To find the value of x is to solve the equation.

Equations are very important instruments in algebra, and a knowledge of their use is the key to the solution of many otherwise difficult problems.

An equation which involves the unknown quantity in the first degree is called a **simple equation.** The following, $x^2 + 3x = 30$, is not a simple equation, because it involves the unknown quantity x to "the square," or the "second power," or the "second degree," as it is called.

It is usual to denote the unknown quantity in equations by the letter x.

The process of solving simple equations depends upon the following **axioms**:

1. *If to equals we add equals, the sums are equal.*
2. *If from equals we take equals, the remainders are equal.*
3. *If equals are multiplied by equals, the products are equal.*
4. *If equals are divided by equals, the quotients are equal.*

Any term may be *transposed* from one side of an equation to the other by changing its sign. Thus:

$$3x - 8 = x + 12.$$

Therefore, $$3x - x = 12 + 8,$$

and $$2x = 20,$$

and $$x = 10.$$

In this example, we carried the x from the right to the left side, changing its sign to a **minus**; and we carried -8 to the right side, changing its sign to a **plus**. The principles upon which these transpositions depend are the first and second axioms above. When we transposed $+x$ from the right side to the left side of the equation and made it

$-x$, we in reality subtracted x from both sides of the equation. When we transposed -8 from the left side to the right side of the equation and made it $+8$, we in reality added 8 to both sides of the equation.

EXERCISES

Find the value of x in the following equations:

1. $12x + 3x = 30.$
2. $12x - 3x = 36.$
3. $9x + 4x = 40 + 12.$
4. $3x - x = 24 - 6.$
5. $19x - 7x = 96 + 48.$
6. $3x - 18 = 7 - 2x.$
7. $3x = 7 - 2x + 8.$
8. $8x - 3 - 5x - 5 = 7x.$
9. $4x = 13 - 2x - 10.$
10. $12x - 8 = 3x + 28.$
11. $24x - 49 = 19x - 14.$
12. $4x - 22 = 34 - 3x.$
13. $26 - 8x = 80 - 14x.$
14. $x + 5 = 3x - 9.$
15. $17x - 114 = 198 - 7x.$

Lesson No. 4. Algebraic Problems

The principles of the last lesson in algebra may now be employed to solve various problems. The method of procedure is as follows:

1. *Let x represent the unknown number or quantity that is to be found.*
2. *Then express in symbolical (algebraical) language the conditions of the equation.*

Problem: Find two numbers whose sum is 28, and whose difference is 4.

We begin by taking x to represent the smaller number. Then $x + 4$ will equal the larger number, and $x + x + 4$, or $2x + 4$, will equal the sum of the numbers, or 28. This can be set down as follows:

Let $\qquad x = \text{smaller number.}$

Then $\qquad x + 4 = \text{larger number.}$

$$2x + 4 = 28.$$

$$2x = 28 - 4.$$

$$2x = 24.$$

$$x = 12, \text{ the smaller number.}$$

$$x + 4 = 16, \text{ the larger number.}$$

The beginner is advised to test his solution by finding whether his result satisfies the conditions of the question or not.

EXERCISES

1. Six times a number increased by 11 is equal to 65. Find the number.

2. Find two numbers whose sum is 39, and whose difference is 5.

3. Divide the number 50 into two parts, such that one part shall lack 10 of being five times the other.

Note. — If x equals one part, then $5x - 10$ equals the other.

4. Divide the number 35 into two parts, so that the greater shall be six times the smaller.

Note. — If x equals smaller part, then $6x$ equals larger.

5. What number added to twice itself makes 72?

6. Find a number such that when 12 is added to its double the sum will be 28.

7. If I add 46 to a certain number, the result will be three times as large as the original number. Find the original number.

8. The difference of two numbers is 54, and their sum is 88. Find the numbers.

9. One number is three times as great as another. If I take the smaller from 16 and the greater from 30, the remainders are equal. Find the numbers.

10. Divide \$124 between A and B, so that A will have \$42 more than B.

Lesson No. 5. Algebraic Subtraction

In algebra the process of subtraction is expressed by the following rule:

Change all signs of the quantities to be subtracted and proceed the same as in addition.

This rule is generally spoken of as "the rule of signs for subtraction." For the sake of convenience like signs should be written under like signs as in addition. Note the following illustration:

$$\begin{array}{ll} \text{From} & 9a^2 + 3ab - 3bc \\ \text{take} & \underline{2a^2 - ab + 4bc} \\ & 7a^2 + 4ab - 7bc \end{array}$$

It is not necessary that the signs be actually changed. The operation of changing the signs should be performed mentally.

The reason for the rule in regard to signs in subtraction is somewhat hard to understand, but it will be apparent after a little thought.

(1) When we subtract $2a^2$ from $9a^2$ it is evident that we must have $7a^2$. That is, the result agrees with the result given us by the rule of signs.

(2) When we subtract $-ab$ from $+3ab$, the rule of signs gives us as result $4ab$. This is not so easy to see through, but it will be understood if we think for a moment. If $+3ab$ is something a *part* of which is $-ab$, it is evident that the *other* part *must* be $+4ab$. Therefore, when we subtract $-ab$ from $+3ab$ the result must be $+4ab$.

(3) When we subtract $+4bc$ from $-3bc$, the rule of signs gives us as result $-7bc$. This is easily seen through. $-3bc$ is less than nothing. If we subtract $+4bc$ from it, we must have something still further less than nothing,

further by $4\,bc$. That is, we must have something less than nothing by $7\,bc$. That is, we must have $-7\,bc$.

Terms containing different powers of the same letter should be put in different columns. For instance, $3\,x^2$ should not be placed under $5\,x$, for it cannot be added to $5\,x$ so as to form one term.

EXERCISES

1. From $a+b+c$ take $-a-b-c$.
2. From $8\,ab-7\,b+7$ take $ab+6\,b-4$.
3. From $4\,a^2+2\,ab$ take $3\,a^2-5\,ab+c$.
4. From $3\,x+y-z$ take $x-4\,y+3\,z$.
5. From $4\,ab-3\,c+a^2$ take $3\,a^2-2\,ab+4\,c$.
6. From $x+8\,y+8\,z$ take $10\,x-7\,y+3\,z$.
7. From $3\,x^2-2\,y$ take $3\,y+4\,xy$.
8. From $x^3+7\,x^2+x$ take $x^3-3\,x^2-2\,x$.
9. From $2\,y-3\,x^2$ take $13+19\,x^2$.
10. From $1-2\,x+3\,x^2$ take $2\,x^2+4\,x+2$.

Note that in writing an algebraical expression it does not matter in what order the terms are written so long as each term carries with it the sign which properly precedes it.

Lesson No. 6. The Removal of Brackets

When in algebra subtraction is to be performed, as, for example, when $3\,a^2-5\,ab+c$ is to be subtracted from $4\,a^2+2\,ab$, what we have to do may be expressed algebraically thus:

$$(4\,a^2+2\,ab)-(3\,a^2-5\,ab+c).$$

From our knowledge of subtraction gained in the previous lesson, we know that this becomes

$$4\,a^2+2\,ab-3\,a^2+5\,ab+c.$$

We have here the explanation of an important algebraical principle: When an algebraical expression is contained in

brackets, and we desire to remove the brackets from it, if the expression be preceded by a plus sign (or a plus sign understood), the signs of each of the terms of the expression remain unchanged; but if the expression be preceded by a minus sign, the signs of each of the terms of the expression are changed. We have thus the following rules for the removal of brackets:

1. *When an expression within brackets is preceded by the sign plus, the brackets can be removed without making any change in the expression.*

2. *When an expression within brackets is preceded by the sign minus, the brackets may be removed if the sign of every term within the brackets be changed.*

ILLUSTRATIVE EXAMPLE

$$\begin{aligned} 2a + (a - b) - (3b + 2a) + a &= 2a + a - b - 3b - 2a + a \\ &= 2a - 4b. \end{aligned}$$

When there are two or more pairs of brackets to be removed, it is generally best to begin with the innermost pair, and then deal with each pair in succession, according to Rules 1 and 2.

ILLUSTRATIVE EXAMPLES

1.
$$\begin{aligned} 4a + [2a - (a - b)] &= 4a + [2a - a + b] \\ &= 4a + [a + b] \\ &= 4a + a + b \\ &= 5a + b. \end{aligned}$$

2.
$$\begin{aligned} 4a - [2a - (a - b)] &= 4a - [2a - a + b] \\ &= 4a - [a + b] \\ &= 4a - a - b \\ &= 3a - b. \end{aligned}$$

EXERCISES

Simplify by removing brackets, and collecting like terms:

1. $a+2b+(2a-3b)$.
2. $2a-3b-(2a+2b)$.
3. $(x-3y)+(2x-4y)-(x-8y)$.
4. $(x-3y+2z)-(z-4y+2x)$.
5. $2a+(b-3a)-(4a-8b)-(6b-5a)$.
6. $a+2b-(2a-3b)$.
7. $a-2-(4-3a)$.
8. $a+2b-3c-(b-a-4c)$.
9. $4x-(2y+2x)-(3x-5y)$.
10. $5x-(7y+3x)-(2y+7x)-(3x+8y)$.

Lesson No. 7. Review

EXERCISES

1. Add: $3ab+bc-ca$; $-ab+ca$; $ab-2bc+5ca$. From the sum take $5ca+bc-ab$.

2. Simplify: $3b-2b^2-(2b-3b^2)$.

3. Simplify: $3a-2b-(2b+a)-(a-5b)$.

4. Subtract $8c^2-8c-2$ from c^3-1.

5. Find the value of x in $2x+3=16-(2x-3)$.

6. Find two consecutive numbers whose sum shall be 79.

7. What expression must be added to $5a^2-3a+12$ to produce $9a^2-7$?

8. To what expression must $3x-4x^3+7x^2+4$ be added to make zero?

9. If $a=1$, $b=3$, $c=2$, $d=0$, find the value of $3a^2-2bc-ad+3b^2cd$.

10. Six times a number increased by 12 is equal to 60. Find the number.

11. Find x in: $6x+3x+9x=108$.

12. Find x in: $9x - 3(5x - 6) = -30$.

13. Divide 180 into two such parts that 5 times one part may be equal to 4 times the other.

14. Find x in: $12(x - 3) - 3(2x - 1) + 5x = 22$.

15. By how much does $b + c$ exceed $b - c$?

Lesson No. 8. Algebraic Multiplication

It has been shown that the factors in a term may be written in any order without affecting the value of the result. But it is usual to write the letters in alphabetical order; that is, the product of a, c, and b may be expressed as acb or abc or cba, etc., but the usual way of writing is abc.

Let us consider the case of the multiplication of two simple algebraical quantities:

To multiply $3ab$ by $4b$.

$$3ab \times 4b = 3 \times a \times b \times 4 \times b = 3 \times 4 \times a \times b \times b = 12ab^2.$$

Rule 1: *Multiply the numerical coefficients together and place the product of the letters in its simplest form after this product.*

Thus: $$3a^2b \times 2ab = 6a^2abb = 6a^3b^2.$$

Rule 2: *In multiplication, like signs give* plus *in the product and unlike signs give* minus *in the product.*

Thus (1): $(+2a) \times (+3b) = +6ab$,

(2): $(-2a) \times (+3b) = -6ab$,

(3): $(+2a) \times (-3b) = -6ab$,

(4): $(-2a) \times (-3b) = +6ab$.

The rule of signs in multiplication in algebra is a little difficult to understand, but its correctness may be inferred after a little thought. When the multiplier is a *plus* quantity, there is no difficulty. We are simply repeating the multiplicand a certain number of times, and therefore the sign of the product will be *the same* as that of the multipli-

cand. There is no reason why it should be changed. But when the multiplier is a *minus* quantity, we are not only repeating the multiplicand a certain number of times, but we are at the same time supposing that it is to be *subtracted* when so repeated. Therefore, according to the rule of subtraction, its sign must be changed; and therefore the sign of the product will be *different* from that of the multiplicand.

The product of three or more expressions is called the **continued product** of these expressions.

ILLUSTRATIVE EXERCISE

Multiply $a^2b - 3a + 2b^2$ by $5ab$.

$$\begin{array}{r} a^2b - 3a + 2b^2 \\ +5ab \\ \hline 5a^3b^2 - 15a^2b + 10ab^3 \end{array}$$

Each term of the expression is taken separately. $(+2b^2) \times (+5ab) = 10abb^2 = 10ab^3$, and so on with each of the other terms.

EXERCISES

Find the value of:

1. $5x \times 7$.
2. $3 \times 2b$.
3. $x^2 \times x^3$.
4. $5x \times 6x^2$.
5. $6c^3 \times 7c^4$.
6. $9y^2 \times y^5$.
7. $4ab \times 3ab$.
8. $6ax \times 2x^2$.
9. $3xy \times 4y^2$.
10. $3ac \times 2ad$.
11. $a^2 \times a^3b \times 5ab^4$.
12. $7a^2 \times 3b^3 \times 4ab$.
13. $2ab \times 2b^2 \times 3b$.
14. $a^2b \times ab^2 \times ab$.
15. $ab - ac$ by a^2c.
16. $2a^2 - 3b$ by $3b$.
17. $3b^2 + 4b$ by $-b$.
18. $a^2 - 2b^3$ by $-3c$.
19. $4b^2c - 16$ by -2.
20. $a^3 - 3a^2x$ by $2a^2bx$.

Lesson No. 9. Algebraic Multiplication (Continued)

To multiply one compound expression by another, multiply each term of the multiplicand by each term of the multiplier, and add results for the complete answer.

To find the product of $a + b$ and $c + d$.

$$\begin{aligned}(a+b) \times (c+d) &= (a+b)\ (c+d)\\ &= a(c+d) + b(c+d)\\ &= ac + ad + bc + bd.\end{aligned}$$

Similarly it may be shown that:

$$\begin{aligned}(a-b)\ (c+d) &= ac + ad - bc - bd,\\ (a+b)\ (c-d) &= ac - ad + bc - bd,\\ (a-b)\ (c-d) &= ac - ad - bc + bd.\end{aligned}$$

These results may be obtained as follows:

1.
$$\begin{array}{l} a + b \\ \underline{c + d} \\ ac + bc \\ \underline{\quad + ad + bd} \\ ac + bc + ad + bd \\ \text{or } ac + ad + bc + bd. \end{array}$$

2.
$$\begin{array}{l} a - b \\ \underline{c + d} \\ ac - bc \\ \underline{\quad + ad - bd} \\ ac - bc + ad - bd \\ \text{or } ac + ad - bc - bd. \end{array}$$

3.
$$\begin{array}{l} a + b \\ \underline{c - d} \\ ac + bc \\ \underline{\quad - ad - bd} \\ ac + bc - ad - bd \\ \text{or } ac - ad + bc - bd. \end{array}$$

4.
$$\begin{array}{l} a-b \\ c-d \\ \hline ac-bc \\ \quad -ad+bd \\ \hline ac-bc-ad+bd \\ \text{or } ac-ad-bc+bd. \end{array}$$

Sometimes, and very frequently, it is possible to arrange the terms of the partial products under one another in such a way as to add easily.

For example, in multiplying $3a-b$ by $2a-3b$.

$$\begin{array}{l} 3a+b \\ 2a-3b \\ \hline 6a^2+2ab \\ \qquad -9ab-3b^2 \\ \hline 6a^2-7ab-3b^2 = \text{product.} \end{array}$$

It does not matter which term of the multiplier is taken first; the result will be the same.

For example, in the above, when we multiply $3a+b$ by $2a-3b$.

$$\begin{array}{r} 3a+b \\ 2a-3b \\ \hline -9ab-3b^2 \\ 6a^2+2ab \qquad \\ \hline 6a^2-7ab-3b^2 = \text{product.} \end{array}$$

It is customary, however, to begin to multiply with the left-hand term of the multiplier.

EXERCISES

Find the product of:

1. $a+3$ and $a+7$.
2. $b-4$ and $b+3$.
3. $3a+2ab$ and $a+b$.

4. $4ab - b^2 + 2$ and $3a + b$.
5. $x^2 - xy + y^2$ and $x - y$.
6. $2a^3 + 4a^2 + 8a + 16$ and $3a - 6$.
7. $a + 2b - 3c$ and $a - 2b + 3c$.
8. $x^2 - 6x - 10$ and $3x^2 - 5x - 5$.
9. $a^3 + a^2 + a$ and $a - 1$.
10. $2a^2 - 3ab - b^2$ and $a + b$.
11. $(a + b)$ by $(a - b)$.
12. $a^3 - b^2 - 3c^2$ by $a + b + c$.
13. $x^2 + x - 2$ by $x^2 - x + 2$.
14. $2a^2 - 3a - 6$ by $a^2 - a + 2$.
15. $a + b - c$ by $a - b + c$.

Write the values of the following:

16. $(a + 3)(a - 2)$.
17. $(a + b)(a - b)(a^2 + b^2)$.
18. $(c - 4)(c + 3)(c - 2)$.
19. $(a + b + c)(a - b - c)$.
20. $(3x + 2a)(4x - 3a)$.

Lesson No. 10. Multiplication by Inspection

Certain results in multiplication should be carefully noticed. Although the product of two binomial factors, such as $x + 3$ and $x - 4$, can always be obtained by the methods already explained, it is of the utmost importance that the student should soon learn to write down such factors rapidly *by inspection.*

Notice the following:

$$(x + a)(x + b) = x^2 + ax + bx + ab = x^2 + x(a + b) + ab,$$
$$(x - a)(x - b) = x^2 - ax - bx + ab = x^2 - x(a + b) + ab,$$
$$(x + a)(x - b) = x^2 + ax - bx - ab = x^2 + x(a - b) - ab,$$
$$(x - a)(x + b) = x^2 - ax + bx - ab = x^2 - x(a - b) - ab.$$

The product of expressions of these forms can therefore be written at once by observing that

1. The product always consists of three terms.
2. The first term of the product is the square of the common term.
3. The second term of the product is the common term multiplied by the sum of the second terms.
4. The third term of the product is the product of the second terms.

Note that when we speak of the sum of the second terms we mean the **algebraic sum**, and when we speak of the product of the second terms we mean the **algebraic product**.

Notice carefully the following illustrations of these statements:

$$\begin{aligned}(x+8)(x+7) &= x^2+(8+7)x+(8\times 7)\\ &= x^2+15x+56,\\ (x-8)(x-7) &= x^2+(-8-7)x+(-8\times -7)\\ &= x^2-15x+56,\\ (x+8)(x-7) &= x^2+(8-7)x+(8\times -7)\\ &= x^2+x-56,\\ (x-8)(x+7) &= x^2+(-8+7)x+(-8\times 7)\\ &= x^2-x-56.\end{aligned}$$

The intermediate step in the work may be omitted and the products written down at once. In a similar way you may write down the product of any two binomials.

NOTE. — "Binomial expressions" are expressions involving only two terms each.

The difficulty, if any, will be with the middle term of the product. The coefficient of x is the *algebraic sum* of the two right-hand terms of the binomials.

EXERCISES

Find by inspection the values of:

1. $(x+2)(x+5)$.
2. $(x+3)(x+7)$.
3. $(x+2)(x+7)$.
4. $(x+4)(x+9)$.
5. $(x+7)(x+5)$.
6. $(x+4)(x+8)$.
7. $(a-2)(a+3)$.
8. $(a-2)(a-3)$.
9. $(a+3)(a+2)$.
10. $(a-5)(a+3)$.
11. $(x-3)(x+12)$.
12. $(x+7)(x-3)$.
13. $(b+2)(b-9)$.
14. $(b-3)(b-4)$.
15. $(c-7)(c+8)$.
16. $(a-1)(a+1)$.
17. $(b-4)(b-4)$.
18. $(a+b)(a-b)$.
19. $(a-b)(a-b)$.
20. $(a-2b)(a+2b)$.
21. $(3x-1)(3x+1)$.
22. $(4x-2)(3x+2)$.
23. $(2x-3)(3x+2)$.
24. $(a+4c)(a-4c)$.
25. $(a+20)(a-10)$.
26. $(x-y)(x-y)$.
27. $(ab-1)(ab-3)$.
28. $(ac+2)(ac+3)$.
29. $(5x-2)(5x-1)$.
30. $(ab-4c)(ab-5c)$.

Lesson No. 11. Simple Equations (Continued)

The student should constantly practice the solving of simple equations until he is thoroughly familiar with the methods involved in a great variety of examples. Note the following illustrations:

(1) Solve $27x-11=127-19x$.

Solution:

Transposing, $27x+19x=127+11$.

Collecting, $46x=138$.

Dividing by 46, $x=3$.

(2) Solve $12(x-3)-3(2x-1)=22-5x$.

Solution: Removing brackets,

$$12x-36-6x+3=22-5x.$$

Transposing, $\quad 12x-6x+5x=22+36-3.$

Collecting, $\quad 11x=55.$

Dividing by 11, $\quad x=5.$

(3) Solve $(x-3)(x-4)-22=(x-5)(x-6)$.

Solution: Removing brackets,

$$x^2-7x+12-22=x^2-11x+30.$$

Transposing, $\quad x^2-x^2-7x+11x=30-12+22.$

Collecting, $\quad 4x=40.$

Dividing by 4, $\quad x=10.$

EXERCISES

Solve the following equations:

1. $(3x+1)(2x-7)=6(x-3)(x-3)+7.$
2. $2(x+1)(x+3)+8=(2x+1)(x+5).$
3. $7(25-x)-2x-15=2(3x-25)-x.$
4. $5x-6(x-4)=2(x+5)+5(x-4)-6.$
5. $(x+3)(2x-3)-6x=(x-4)(2x+4)+12.$
6. $(x+2)(x+3)+(x-3)(x-2)-2x(x+1)=0.$
7. $8(x-3)-2(3-x)=2(x+2)-5(5-x).$
8. $4(5-x)-2(x-3)=x-4-3(x+2).$
9. $5-4(x-3)=x-2(x-1).$
10. $(2x+1)(2x+6)-7(x-2)=4(x+1)(x-1)-9x.$

Solve the following problems:

11. The sum of two numbers is 20, and if three times the smaller be added to five times the greater the sum is 84. Find the number.

12. The sum of two numbers is 100, and twice the less exceeds the greater by 5. Find the numbers.

13. A having three times as much as B, gives B \$2. A then has twice as much as B. How much had each at first?

14. Divide \$1000 among A, B, and C, so that B shall have \$100 more than A, and A four times as much as C. (Let $x =$ C's share.)

15. A house and garden cost \$3000; 10 times the price of the house was equal to 50 times the price of the garden. Find the price of each.

16. A father is three times as old as his son; four years ago he was four times as old as the son then was. What is the present age of each?

17. Divide 60 into two parts, so that three times the greater exceeds 100 by as much as eight times the less falls short of 200.

NOTE. — Let $x =$ the greater part; then $60 - x =$ the less.

Three times the greater part is $3x$, and its excess over 100 is $3x - 100$.

Eight times the less is $8(60 - x)$, and its defect from 200 is $200 - 8(60 - x)$.

Therefore the equation is

$$3x - 100 = 200 - 8(60 - x).$$

18. Divide 122 into two parts, such that when one part is decreased by 2 the other part may be twice as great as the remainder.

19. The difference between the squares of two consecutive numbers is 31. Find the numbers.

20. Find the number whose double exceeds its half by 9.

Lesson No. 12. Algebraic Division

Division in algebra is very similar to division in arithmetic.

The operation is denoted by the sign $\div$.

Since the product of 3 and x is $3x$, it follows that when $3x$ is divided by x the quotient is 3.

Thus, $\quad ab \div b = a$, and $a^2 \div a = a$.

The operation may be denoted as in arithmetic by writing the dividend over the divisor; as,

$$3a^2b \div a = \frac{3a^2b}{a} = 3ab.$$

We remove from the divisor and dividend the factors common to both, just as in arithmetic. The index of any letter in the quotient is the difference of the indices of that letter in the dividend and divisor.

ILLUSTRATIVE EXERCISE

$$21a^3b^2c \div 3a^2b = \frac{21a^3b^2c}{3a^2b} = \frac{3 \times 7 \times a \times a \times a \times b \times b \times c}{3 \times a \times a \times b}.$$

Then, by canceling the like factors in each,

$$= 7 \times a \times b \times c = 7abc.$$

In division, as in multiplication, like signs produce *plus* and unlike signs produce *minus*.

Knowing the rule of signs in multiplication, it is easy to prove that it holds good in division. Thus:

$+ab \div +a = \dfrac{ab}{a} = \dfrac{a \times b}{a} = b$, by cancellation.

$-ab \div -a = \dfrac{-ab}{-a} = \dfrac{-a \times b}{-a} = b$, by cancellation.

$+ab \div -a = \dfrac{ab}{-a} = \dfrac{(-a) \times (-b)}{-a}$, by the rule of signs in multiplication,

$= -b$, by cancellation.

$-ab \div +a = \dfrac{-ab}{a} = \dfrac{a \times (-b)}{a}$, by the rule of signs in multiplication,

$= -b$, by cancellation.

In the above four results it is observable that when the signs of the dividend and divisor are alike, the sign of the quotient is *plus*; but that when the signs of the dividend and divisor are unlike, the sign of the quotient is *minus*.

EXERCISES

1. $abc \div c.$
2. $3\,ab \div b.$
3. $4\,b^2 \div 2\,b.$
4. $5\,bc \div bc.$
5. $3\,a^2b \div ab.$
6. $9\,a^3b \div 3\,a^2.$
7. $7\,ab^2 \div ab.$
8. $6\,abc \div 6.$
9. $8\,b^2c^2 \div 2\,bc.$
10. $10\,abc \div 5\,a.$
11. $-3\,a^2 \div a.$
12. $-4\,ab \div 2.$
13. $-5\,ab^2 \div -5.$
14. $-9\,bc \div 3\,b.$
15. $27\,ab \div -9.$
16. $12\,b^2 \div -3\,b.$
17. $4\,ab \div -2.$
18. $18\,b^3c^2 \div 3\,b^2c.$
19. $15\,a^4b^2 \div 5\,a^3b.$
20. $-12\,a \div -12.$

To divide a compound expression by a single factor, divide each term separately by that factor and take the algebraic sum of the partial quotients so obtained.

21. $(63\,a^2 + 21\,b) \div 7.$
22. $(42\,ab + 6\,b) \div 3\,b.$
23. $(7\,a^2 - ab) \div a.$
24. $(8\,ab - 12) \div 4.$
25. $(10\,a^3 - 5\,a^2) \div 5\,a.$
26. $(x^2 - xy) \div x.$
27. $(12\,ab - a^2b^2) \div ab.$
28. $(4\,a^3 - a^2) \div a^2.$
29. $(9\,b^3 + 9) \div 3.$
30. $(3\,x^2 - 2\,x) \div x.$

Lesson No. 13. Algebraic Division (Continued)

EXERCISES

1. Divide $a^5 + a^4 + a^3 + a^2$ by a^2.
2. Divide $8\,a^4 - 6\,a^3 - 10\,a^2 + 12\,a$ by $2\,a$.
3. Divide $48\,a^2b - 36\,ab^2 + 12\,a^2b^2$ by $6\,ab$.
4. Divide $3\,a^3b^2c + 6\,ab^3c^3 - 9\,abc$ by $3\,ab$.
5. Divide $a^2xy - 2\,axy^2$ by ay.
6. Divide $3\,x^3 - 6\,x^5 + 9\,x^7 - 12\,x^4$ by $3\,x^3$.
7. Divide $a(a+b) + b(a+b)$ by $a+b$.
8. Divide $(a+b)^2 + (a+b)$ by $a+b$.
9. Divide $-x^3z^3 - 3\,xz + x^2z^2$ by $-xz$.
10. Divide $3\,a(a+b) + 9\,a^2(a+b)$ by $3\,a$.

Lesson No. 14. Algebraic Division (Continued)

In the division of one compound expression by another, the following rules should be noted:

1. *Arrange the divisor and the dividend in ascending or descending powers of some common letter.*

$$11x^2 + 3x + x^3 \div x^2 + 1 - x$$

should be arranged

$$x^3 + 11x^2 + 3x \div x^2 - x + 1.$$

2. *Divide the term on the left of the dividend by the term on the left of the divisor, and put the result in the quotient.*

3. *Multiply the whole divisor by the quotient, and put the product under the dividend.*

4. *Subtract and bring down from the dividend as many terms as may be necessary, using the remainder obtained and the terms thus brought down as a new dividend.*

5. *Repeat these operations until all the terms from the dividend have been brought down.*

ILLUSTRATIVE EXERCISE

Divide $6x^2 + 19x + 10$ by $3x + 2$.

$$\begin{array}{rl} 3x+2)\,6x^2 + 19x + 10\,(2x+5 & \\ \underline{6x^2 + 4x} & \\ 15x + 10 & \\ \underline{15x + 10} & \end{array}$$

Divide $3x$ into $6x^2$, and we have $2x$, the first term of the quotient. Multiply $3x+2$ by $2x$, and the product is $6x^2+4x$. Subtract and bring down the next term, and we have $15x+10$ for a new dividend. $3x$ will divide $15x$ *five* times. Place $+5$ in the quotient and multiply $3x + 2$ by 5, and we have $15x + 10$. $15x + 10$ subtracted from $15x + 10$ leaves no remainder and the division is complete.

The reason for the process will be seen if we separate

$6x^2+19x+10$ into the two parts $6x^2+4x$ and $15x+10$. Now each of these parts is divided by $3x+2$. Thus we obtain the partial quotients $+2x$ and $+5$.

EXERCISES

1. Divide a^2+5a+6 by $a+2$.
2. Divide $x^2+7x+12$ by $x+4$.
3. Divide a^2+2a+1 by $a+1$.
4. Divide x^2+4x+3 by $x+1$.
5. Divide $a^2+16a+63$ by $a+7$.
6. Divide $x^2+ax-6a^2$ by $x+3a$.
7. Divide $m^2+7m-78$ by $m-6$.
8. Divide $x^2-23x+120$ by $x-15$.
9. Divide $12a^2-5a-2$ by $4a+1$.
10. Divide $8a^2+34ab+21b^2$ by $4a+3b$.
11. Divide $x^2+23x+102$ by $x+17$.
12. Divide $a^2b^2+3ab-4$ by $ab-1$.
13. Divide x^6+24x^3+144 by x^3+12.
14. Divide $8x^3-12x^2-14x+21$ by $2x-3$.
15. Divide $12a^2-31ab+20b^2$ by $4a-5b$.

Lesson No. 15. Review

EXERCISES

1. If $x=3$, $y=-2$, $z=0$, find the value of

$$\frac{3x^2+5yz+4y^3}{x-4y+z}.$$

2. Divide $3a^5-16a^4-33a^3-14a^2$ by a^2-7a.

3. Find the product of $2a-3b-(a-2b-c)$ and $b-2c-(a-c)$.

4. Subtract $2b^2-2$ from $-2b+6$, and increase the result by $3b-7$.

5. What expression must be added to $5a^2-3a+12$ to produce $9a^2-7$?

6. Find the value of

$$6ax+(2by-cz)-(2ax-3by+4cz)-(cz+ax),$$

when $a=0,\ b=1,\ c=2,\ x=8,\ y=3,\ z=4.$

7. Divide $12a^2+ax-6x^2$ by $3a-2x$.

8. Subtract $1-x^2$ from 1 and add the result to $2y-x^2$.

9. Simplify $(a+2b-3c)+(b-3a+2c)-(3b-2a-2c)$.

10. Divide $x^4-4x^3-18x^2-11x+2$ by x^2-7x+1.

Lesson No. 16. Factoring

The process known in algebra as **factoring** is the opposite of multiplication. By multiplication we find the product of two or more **factors**; by factoring we find the factors of a given **product** or quantity.

A factor of a quantity is an **exact divisor** of it.

The simplest sort of factoring is that performed when all the terms have a common factor.

RULE: *Divide the quantity by the highest factor or divisor common to all the terms. The divisor so chosen and the quotient will be the required factors.*

Find the factors of $12a^2b-18ab^2$

$$6ab\,\underline{|\,12a^2b-18ab^2}$$
$$2a-3b$$

Therefore the expression when resolved into factors

$$=6ab(2a-3b).$$

EXERCISES

Resolve into factors:

1. a^2+ab.
2. a^3+2a^2b.
3. $4am+8a$.
4. $3x^2-6xy$.
5. $5abc-12bc$.
6. $9a^2b^2c^2+3abc$.
7. $4a-a^2$.
8. $3+6x^2$.

9. $24a - 12bc$.

10. $12a^3b + 24a^2$.

11. $x^3 + x$.

12. $4b^2c + 2bc^2 - 8bc$.

13. $3am - 6m + 9a^2m^2$.

14. $42a^2 - 14a + 21$.

15. $a^2(a+b) - 5(a+b)$.

NOTE. — In this example consider $(a + b)$ as a factor of each term. Divide the quantity by $(a + b)$, and we have $(a + b)(a^2 - 5)$.

16. $a(a-b) + 7(a-b)$.

17. $4ab(x+1) + 8a(x+1)$.

NOTE. — Divide each of the terms by $(x + 1)$, and we have $(4ab + 8a)(x + 1)$. Now reduce $(4ab + 8a)$ to factors, and we have $4a(b + 2)(x + 1)$ as the factors of the quantity.

18. $3a^2(b+c) + 6a^3(b+c)$.

19. $4ab(x+y) - 8a^2b(x+y)$.

20. $x(x+1) - 5x^2(x+1)$.

Lesson No. 17. Factoring (Continued)

We have already learned how to factor expressions in which the terms contain a compound factor; as,

$$a(a+b) + c(a+b) = (a+c)(a+b).$$

If we remove the brackets from the original expression, we have $a^2 + ab + ac + cb$. We shall learn in this lesson how to factor expressions consisting of four terms changeable into expressions, each term of which contains a common compound factor.

1. Resolve into factors: $x^2 + ax + bx + ab$.

SOLUTION

Since the first two terms contain a common factor x and the second two terms contain a common factor b, we have

$$\begin{aligned} x^2 + ax + bx + ab &= (x^2 + ax) + (bx + ab) \\ &= x(x+a) + b(x+a) \\ &= (x+a)(x+b). \end{aligned}$$

2. Resolve into factors: $9ab + 12bc - 6ad - 8cd$.

SOLUTION

$$\begin{aligned}9\,ab + 12\,bc - 6\,ad - 8\,cd &= (9\,ab + 12\,bc) - (6\,ad + 8\,cd)\\ &= 3\,b\,(3\,a + 4\,c) - 2\,d\,(3\,a + 4\,c)\\ &= (3\,b - 2\,d)\,(3\,a + 4\,c).\end{aligned}$$

Notice that in arranging the expression in the form of two compound terms it is necessary to change the sign in $-8\,cd$ to $+8\,cd$ on account of the *minus* sign preceding the bracket.

3. Resolve into factors: $ac - ad - bc + bd$.

SOLUTION

$$\begin{aligned}ac - ad - bc + bd &= a\,(c - d) - b\,(c - d)\\ &= (a - b)\,(c - d).\end{aligned}$$

With a little practice the student should be able to write down the factors without doing the work step by step as shown above.

EXERCISES

Resolve into factors:

1. $ax + by + bx + ay$.
2. $x^3 + x^2 + ax + a$.
3. $9 + 3\,x + 3\,y + xy$.
4. $x^2 + xy + xz + yz$.
5. $a^2 + 2\,a + ab + 2\,b$.
6. $2\,a + 2\,x + ax + x^2$.
7. $am - bm - an + bn$.
8. $ax - 2\,ay - bx + 2\,by$.
9. $ac^2 + b + bc^2 + a$.
10. $a^2 + ac + 4\,a + 4\,c$.
11. $2\,mx + nx + 2\,my + ny$.
12. $ac^2 - 2\,a - bc^2 + 2\,b$.
13. $axy + bcxy - ad - bcd$.
14. $12a^2 - 18ab + 8ac - 12bc$.
15. $abx^2 - axy + bxy - y^2$.

If the terms of the expression are not in convenient order, they may, of course, be changed to any other order that is convenient.

Lesson No. 18. Factoring (Continued)

To find the factors of $x^2 + 7\,x + 12$.

We must find two numbers whose product is 12 and whose sum is 7. These are 3 and 4.

Then the factors of $x^2+7x+12$ are $(x+3)(x+4)$.

This is a particular case of the general principle established in Lesson 10.

EXERCISES

Factor the following:

1. x^2+5x+6.
2. x^2+6x+9.
3. x^2+7x+6.
4. $x^2+8x+16$.
5. x^2+4x+4.
6. $x^2+9x+18$.
7. $x^2+7x+10$.
8. $x^2+9x+20$.
9. $x^2+10x+25$.
10. $x^2+10x+21$.
11. $a^2+7a+12$.
12. $a^2+15a+56$.
13. $a^2+10a+24$.
14. $a^2+12a+32$.
15. $b^2c^2+3bc+2$.
16. $a^2b^2+7ab+10$.
17. a^4+10a^2+25.
18. a^6+7a^3+12.
19. b^2+2b+1.
20. a^2+6a+9.

Lesson No. 19. Factoring (Continued)

To find the factors of $x^2-7x+12$.

We must find two numbers whose product is $+12$ and whose algebraic sum is -7. These are evidently -3 and -4.

Therefore the factors of $x^2-7x+12$ are $(x-3)(x-4)$.

NOTE. — Since the third term of $x^2-7x+12$ is *plus*, it is evident that the second terms of factors will have their signs *both alike;* and since the second term of the expression is *minus*, the second terms of its factors will be both *minus*.

The process of factoring here exemplified is a particular case of the general principle established in Lesson 10.

To find the factors of $x^2+3x-10$.

We must find two numbers whose product is -10 and whose algebraic sum is $+3$. These are evidently $+5$ and -2.

Therefore the factors of $x^2 + 3x - 10$ are

$$(x + 5)(x - 2).$$

NOTE. — Since the third term of $x^2 + 3x - 10$ is *minus*, it is evident that the second terms of its factors will have their signs *unlike;* and since the second term of the expression is *plus*, the greater of the terms of its factors will be *plus.*

The process of factoring here exemplified is also a particular case of the general principle established in Lesson 10.

EXERCISES

Factor the following:

1. $x^2 + 5x - 14.$
2. $x^2 - 5x - 14.$
3. $x^2 - 6x - 7.$
4. $m^2 - 10m + 9.$
5. $b^2 - 3b + 2.$
6. $x^2 - 7x + 12.$
7. $n^2 + n - 2.$
8. $x^2 - 7x - 18.$
9. $x^2 + 4x - 12.$
10. $a^2 - 7a + 10.$
11. $a^2 - 5a - 24.$
12. $a^2 + 5a - 36.$
13. $x^2 - 10x - 200.$
14. $a^2 - 14a + 40.$
15. $m^2 + 10m + 24.$
16. $x^2 - 14x + 48.$
17. $a^2 + a - 420.$
18. $x^2 + 2x - 255.$
19. $x^2 - 9x + 18.$
20. $a^2b^2 - ab - 20.$

Lesson No. 20. Factoring (Continued)

In finding the factors of expressions of three terms, such as we have been examining (namely, those in which the coefficient of the highest power is unity), the difficulty will generally be with the signs of the second terms of the factors. When the signs of the second and third terms of the expression to be factored are *both plus*, the connecting signs of the factors must *both be plus.* When the sign of the second term of the expression is *minus* and that of the third term *plus*, the connecting signs of the factors must *both be minus.* When the sign of the *third* term of the ex-

pression is *minus*, the connecting signs of the factors will be one *plus* and the other *minus*, and the second terms must be such that their algebraic sum equals the coefficient of the second term of the expression.

Notice carefully the following illustrations:

1. Find the factors of $x^2 + 5x - 84$.

Here we must find two terms, one positive and one negative, whose product is -84 and whose algebraic sum is 5. These are evidently 12 and -7. Therefore the factors are $(x + 12)$ and $(x - 7)$.

2. Find the factors of $x^2 - 3x - 28$.

Here we must find two terms, one positive and one negative, whose product is -28 and whose algebraic sum is -3. These are evidently 4 and -7. Therefore the factors are $(x + 4)$ and $(x - 7)$.

3. Find the factors of $x^2 - 12x + 36$.

Here we must find two terms, both negative, whose product is 36 and whose algebraic sum is -12. These are -6 and -6. Therefore the factors are $(x - 6)$ and $(x - 6)$, or $(x - 6)^2$.

EXERCISES

Resolve into factors:

1. $x^2 - 7x + 10$.
2. $x^2 - 29x + 190$.
3. $x^2 - 23x + 132$.
4. $a^2 - 30a + 200$.
5. $a^2 - 43a + 460$.
6. $x^2 - 57x + 56$.
7. $b^2 - 7b + 12$.
8. $x^2 - 9x + 20$.
9. $x^2 + 7x - 60$.
10. $x^2 + 12x - 45$.
11. $a^2 + 11a - 12$.
12. $a^2 + 13a - 140$.
13. $b^2 + 13b - 300$.
14. $m^2 + 3m - 4$.
15. $n^2 + 17n - 390$.
16. $x^2 - 5x - 66$.
17. $x^2 - 7x - 18$.
18. $m^2 - 9m - 36$.
19. $n^2 - 11n - 60$.
20. $a^2 - 13a - 14$.
21. $b^2 - 15b - 100$.
22. $x^2 - 15x + 36$.

23. $x^2 + 4x - 45$.

24. $a^2b^2 - 16ab - 36$.

25. $a^2 + a - 90$.

26. $a^2 + 12a + 36$.

27. $a^2 + 6a - 7$.

28. $a^2 + a - 12$.

29. $a^2 - 4a - 117$.

30. $x^2 + x - 132$.

Lesson No. 21. Review

EXERCISES

1. Subtract $3x - 15y - 37z$ from $13y - 11z - 19x$.

2. Divide $x^4 + 24x + 55$ by $x^2 - 4x + 11$.

3. Solve: $3x - 20 + x = 44 - 4x$.

4. Solve: $4x + 4 - 3x = 16 - 2x$.

5. Simplify: $a(a - 3b) + a^2 - 3a(a - b)$.

6. Solve: $7(8 - 3x) + 6(2x - 5) = -28$.

7. When $a = -1$, $b = 3$, $c = -2$, find the value of $2a^3 + b^3 + c^3 - 3abc$.

8. Divide the product of $x^2 - 4$ and $x^2 - 4x$ by $x^2 + 2x$.

9. Find what value of x will make $3(x - 1)$ equal to the sum of $6(x - 2)$ and $5(x - 3)$.

10. Multiply $ab + 2ac - 2bc + a^2 + b^2 + 4c^2$ by $a - b - 2c$.

11. Find the factors of $a^4 + 9a^2 + 20$.

12. Find the factors of $x^2 - 2x + ax - 2a$.

13. Find the factors of $x^2 - 3x - ax + 3a$.

14. Find the factors of $x^2 - 3x - 108$.

15. If I add 17 to the square of a certain number, I obtain the square of the next highest number. What is the number?

Lesson No. 22. Solution of Equations by Factoring

Some equations may be solved by factoring. Note the following illustration:

To find the value of x in $x^2 + 21 = 25$.

SOLUTION

$$x^2 + 21 = 25,$$
$$x^2 = 25 - 21,$$
$$x^2 = 4,$$
$$x = 2.$$

Note that the value of x is either $+2$ or -2. Either multiplied by itself will produce $+4$. As a rule it is only the plus value obtained in the solution of equations of this sort that is of practical importance.

EXERCISES

Find the value of x in the following:

1. $x^2 - 9 = 16.$
2. $x^2 + 4 = 20.$
3. $x^2 + 6 = 70.$
4. $x^2 - 14 = 50.$
5. $x^2 + 1 = 50.$
6. $3x^2 + 4 = 2x^2 + 13.$
7. $x^2 - 8 = 8.$
8. $3x^2 + 2 = 66 - x^2.$
9. $x^2 - 15 = 10.$
10. $x^2 + 36 = 100.$
11. $x^2 + 4 = 40.$
12. $x^2 + 15 - 8 = 8.$
13. $x^2 - 25 = 0.$
14. $x^2 + 16 = 32.$
15. $2x^2 + 2 = 100.$
16. $2x^2 + 2 = x^2 + 11.$
17. $3x^2 - 8 = 4x^2 - 44.$
18. $x^2 + 1 = 82.$
19. $x^2 + 12 = 48.$
20. $3x^2 - 32 = 2x^2 + 112.$

Lesson No. 23. Solution of Equations by Factoring (Continued)

To find the value of x in $x^2 + 6x + 9 = 25$.

SOLUTION

$$x^2 + 6x + 9 = 25.$$

The factors of $x^2 + 6x + 9$ are $(x + 3)$ and $(x + 3)$, or the square root of the expression. The square root of 25 is 5; then since $(x + 3) \times (x + 3) = 5 \times 5$, it follows that

$x+3=5$, for it is evident that if two quantities are equal, then their square roots must be equal. Therefore,

$$x=5-3;$$
$$x=2.$$

EXERCISES

Find the value of x in the following equations:

1. $x^2+4x+4=36$.
2. $x^2+6x+9=49$.
3. $x^2+8x+16=64$.
4. $x^2+10x+25=81$.
5. $x^2+12x+36=100$.
6. $x^2-8x+16=81$.
7. $x^2-16x+64=9$.
8. $x^2+12x+36=49$.
9. $x^2-20x+100=25$.
10. $x^2-14x+49=9$.

Lesson No. 24. Review

EXERCISES

1. Simplify: $14x-(5x-9)-\{4-3x-(2x-3)\}$.

2. Add:
$$-12x-5y+4z;$$
$$3x+2y-3z;$$
$$9x-3y+z.$$

3. From $a+b+c-7$ take $8-c-b+a$.

4. Multiply x^2+xy+y^2 by x^2-xy+y^2.

5. Find the continued product of $x-a$, $x+a$, x^2+a^2, and x^4+a^4.

6. Divide x^3+2x^2+2x+1 by $x+1$.

7. Divide $x^4+x^2y^2+y^4$ by x^2-xy+y^2.

8. Find the factors of

(1) $x^4-ax^3+bx^2+cx$.

(2) $49y^2-14y+7$.

(3) $bm+mn+ab+an$.

(4) $m^2+23m+102$.

(5) $x^2-29x+190$.

9. Find the value of $x^3+y^3-z^3+3xyz$, when $x=3$, $y=2$, $z=5$.

10. What value of x will make the difference between $(2x+4)(3x+4)$ and $(3x-2)(2x-8)$ equal 192?

11. Find the value of x in

(1) $7x+5=5x+11$.

(2) $12x+7=8x+15$.

(3) $5x-7=3x+7$.

(4) $124x+19=112x+43$.

(5) $18-2x=27-5x$.

12. To the double of a certain number I add 14, and obtain as a result 154. What is the number?

13. To four times a certain number I add 16, and obtain as a result 188. What is the number?

14. By adding 46 to a certain number I obtain as a result a number three times as large as the original number. Find the original number.

15. One number is three times as large as another. If I take the smaller from 16 and the greater from 30, the remainders are equal. What are the numbers?

Lesson No. 25. Factoring (Continued)

We proceed now to the resolution into factors of trinomial expressions in which the coefficient of the highest power is not unity.

Note the following:

$$(3x+2)(4x+5)=12x^2+23x+10.$$

It is evident that the middle term of the product is found as follows:

$$\left.\begin{aligned}3x+2&=+8x\\4x+5&=+15x\end{aligned}\right\}=+23x.$$

The converse problem, that is, the *factoring* of an expression like $12x^2+23x+10$, presents some difficulty. The beginner will find that it is not an easy matter to select the proper factors at the first trial. Practice alone will enable him to detect at a glance whether any pair he has chosen will combine so as to give the correct coefficient of the middle term of the expression to be factored.

Example: Factor $15x^2 + 29x + 12$.

Write down *first* trial factors as follows:

$$(3x + 3);$$
$$(5x + 4).$$

By cross multiplication we have:

$$\left.\begin{aligned} 3x + 4 = 12x \\ 5x + 3 = 15x \end{aligned}\right\} = 27x.$$

It is evident that these are not the factors. The sum of the cross multiplication products should be $29x$.

Write down *second* trial factors as follows:

$$(3x + 4);$$
$$(5x + 3).$$

By cross multiplication the products are $20x$ and $9x$. These added give $29x$, the middle term of the expression to be factored. The product of $3x$ and $5x$, or $15x^2$, is the first term, and the product of 4 and 3, or 12, is the third term. We have found, therefore, that the factors of

$$15x^2 + 29x + 12 \text{ are } (3x + 4)(5x + 3).$$

In actual practice it is not necessary to put down all these steps at length. The student will soon find that the different cases may be rapidly inspected and the unsuitable combinations rejected at once.

EXERCISES

Resolve into factors:

1. $6x^2 + 17x + 12$.
2. $6x^2 + 15x + 9$.
3. $8x^2 + 8x + 2$.
4. $15x^2 + 27x + 12$.
5. $14x^2 + 41x + 15$.
6. $12x^2 + 29x + 14$.
7. $25x^2 + 40x + 15$.
8. $28x^2 + 56x + 28$.
9. $6x^2 + 38x + 56$.
10. $15x^2 + 62x + 63$.
11. $8x^2 + 44x + 56$.
12. $72x^2 + 87x + 21$.

13. $7x^2 + 15x + 2$.

14. $16x^2 + 44x + 30$.

15. $12x^2 + 30x + 12$.

16. $63x^2 + 88x + 20$.

17. $24x^2 + 67x + 8$.

18. $4x^2 + 9x + 5$.

19. $21x^2 + 76x + 20$.

20. $16x^2 + 66x + 8$.

Lesson No. 26. Review

EXERCISES

Simplify the following:

1. $a - b - (a + b - c - 3)$.
2. $a - [2b - (3c + 2b) - a]$.
3. $(x^2 + 2xy + y^2) - (2xy - x^2 - y^2)$.
4. $8a - (6a - 5) - (5a + 11 - 4a)$.
5. $2x - (x - 5x + 3x - 8x)$.

Find the value of x in

6. $10x - 22 = 17 - 3x$.
7. $3x - 19 = 20 - 10x + 13$.
8. $9x + 3 - 24 = 5x - 25$.
9. $18x + 9 = 15x + 30$.
10. $10x - 4 + 3 = 6x + 19$.

Find the product of

11. $x - y$ and $x + y$.
12. $2a - 3b$ and $4a + 5b$.
13. $3a^2 + ab - b^2$ by $2a + 3b$.
14. $a^2 - 2ax + 4x^2$ by $a^2 + 2ax + 4x^2$.
15. $8a^3 + 4a^2b + 2ab^2 + b^3$ by $2a - b$.

Divide

16. $x^3 - 8$ by $x - 2$.
17. $a^3 + b^3$ by $a + b$.
18. $10x^2 + 14x - 12$ by $2x + 4$.
19. $x^2 - 4x + 4$ by $x - 2$.
20. $36 + x^4 - 13x^2$ by $6 + x^2 + 5x$.

Solve the following equations:

21. $5(x+1)+6(x+2)=6(x+7)$.
22. $3(x+1)+4(x+2)=6(x+3)$.
23. $x(x+5)-6=x(x-1)+12$.
24. $x^2+16x+64=121$.
25. $a^2+30a+225=625$.

Lesson No. 27. Factoring (Continued)

We now proceed to the most important case of factoring, namely, that in which the expression to be resolved can be put in the form of *two squares*, with a negative sign between them.

Since it is evident by multiplication that

$$(a+b)(a-b)=a^2-b^2,$$

therefore, conversely,

$$a^2-b^2=(a+b)(a-b).$$

It follows then that we can express the difference between the squares of two quantities by the product of two factors. To arrive at these factors we proceed thus:

Take the square root of each term.

The sum of the square roots will give the first factor.

The difference of the square roots will give the second factor.

For example, let x^2-y^2 be the expression.

The square root of x^2 is x.

The square root of y^2 is y.

The sum of the square roots is $x+y$.

The difference is $x-y$.

The factors are therefore $(x+y)$ and $(x-y)$.

That is, $x^2-y^2=(x+y)(x-y)$.

The same method holds good with respect to compound quantities:

Thus, let $a^2 - (b - c)^2$ be the expression.

The square root of the first term is a.

The square root of the second is $(b - c)$.

The sum of the square roots is $a + b - c$.

The difference is $a - (b - c)$ or $a - b + c$.

The factors are therefore $(a + b - c)$ and $(a - b + c)$.

That is, $a^2 - (b - c)^2 = (a + b - c)(a - b + c)$.

Again, let $(a - b)^2 - (c - d)^2$ be the expression.

The factors are $(a - b) + (c - d)$ and $(a - b) - (c - d)$, or $(a - b + c - d)$ and $(a - b - c + d)$.

The terms of an expression may often be arranged so as to form two squares with the negative sign between them, and then the expression can be resolved into factors.

Thus, let $a^2 + 2\,ab + b^2 - c^2$ be the expression.

The factors of $a^2 + 2\,ab + b^2 = (a + b)(a + b) = (a + b)^2$.

Therefore, $a^2 + 2\,ab + b^2 - c^2 = (a + b)^2 - c^2$.

And the factors are $(a + b + c)(a + b - c)$.

Or $a^2 + 2\,ab + b^2 - c^2 = (a + b + c)(a + b - c)$.

EXERCISES

Resolve into factors:

1. $m^2 - n^2$.
2. $a^2 - 9$.
3. $x^2 - 4$.
4. $b^2 - c^2$.
5. $4\,x^2 - 25$.
6. $36\,a^2 - 49\,b^2$.
7. $a^4 - b^2$.
8. $m^4 - 16$.
9. $4\,a^2 - 9\,b^2$.
10. $x^2 - 1$.
11. $(a - b)^2 - d^2$.
12. $x^2 - (m + n)^2$.
13. $(a + b)^2 - c^2$.
14. $x^2 - 2\,xy + y^2 - m^2$.
15. $(a - b)^2 - 16$.
16. $a^2 - 2\,ab + b^2 - 49$.
17. $(5\,x - 2)^2 - (x - 4)^2$.
18. $c^2 - a^2 - 2\,ab - b^2$.
19. $2\,xy - y^2 - x^2 + 1$.
20. $49 - (x + 1)^2$.

Lesson No. 28. Illustrations of the Uses of Factoring

The principles of factoring exemplified in our lessons will be found of immense importance in every part of subsequent algebraical study. As illustrations of how these principles are used, we take up in this lesson two important algebraical operations: (1) the process of finding the highest common factor of two or more algebraical expressions; and (2) the process of reducing an algebraical fractional expression to its lowest terms.

1. *To find the highest common factor of two or more algebraical expressions.*

An expression is said to be a **common factor** of two or more other expressions when each of the other expressions is divisible by the first expression; thus,

$3a$ is a common factor of $12a$ and $15a$.

$4b$ is a common factor of $8b^2$ and $8b$.

x is a common factor of x^2 and $3x$.

The **highest common factor** (or H. C. F.) of two or more expressions is the largest factor, that is to say the factor of highest powers, by which each of the expressions is divisible.

ILLUSTRATIVE EXAMPLE

To find the highest common factor of

$$3a^2 + 9ab;\ a^3 - 9ab^2;\ a^3 + 6a^2b + 9ab^2.$$

Resolve each expression into its factors:

$$3a^2 + 9ab = 3a(a + 3b).$$
$$a^3 - 9ab^2 = a(a + 3b)(a - 3b),$$
$$a^3 + 6a^2b + 9ab^2 = a(a + 3b)(a + 3b).$$

Therefore the H. C. F. is $a(a + 3b)$.

EXERCISES

I

Find the highest common factor of

1. $a^2 + ab$ and $a^2 - ab$.
2. $a^2 + a - 12$ and $a^2 - 2a - 3$.
3. $3(a - b)^3$ and $a^2 - 2ab + b^2$.
4. $ax + x$ and $a^2x + ax$.
5. $a^2 + ay$ and $ay + y^2$.
6. $6(a + 1)^2$ and $9(a^2 - 1)$.
7. $a^2 + 5a + 4$ and $a^2 + 2a - 8$.
8. $2x^2 - 9x + 4$ and $3x^2 - 7x - 20$.
9. $2x^2 - 8x + 8$ and $(x - 2)^2$.
10. $3a^2 + 7a - 6$ and $2a^2 + 7a + 3$.
11. $x^2 + 7x + 12$ and $x^2 + 9x + 20$.
12. $x^2 - 17x + 70$ and $x^2 - 13x + 42$.
13. $x^2 + x - 12$ and $x^2 - 2x - 3$.
14. $x^2 + 5x + 6$.
    $x^2 + 7x + 10$.
    $x^2 + 12x + 20$.
15. $a^2 + 5a + 4$.
    $a^2 + 2a - 8$.
    $a^2 + 7a + 12$.

2. *To reduce an algebraical fractional expression to its lowest terms.*

An algebraical fraction may be reduced to an equivalent fraction by dividing numerator and denominator by a common factor. As in arithmetic, if this factor be the highest common factor, the resulting fraction is said to be in its lowest terms.

ILLUSTRATIVE EXAMPLE

To reduce $\dfrac{6x^2 - 8xy}{9xy - 12y^2}$ to its lowest terms.

$$\text{The fraction} = \frac{2x(3x - 4y)}{3y(3x - 4y)}$$
$$= \frac{2x}{3y}.$$

That is, $\dfrac{2x}{3y}$ is the equivalent of $\dfrac{6x^2 - 8xy}{9xy - 12y^2}$, and in its lowest terms.

The beginner should be careful not to begin canceling until he has expressed both numerator and denominator in the most convenient form, by resolution into factors where necessary.

EXERCISES

II

Reduce to equivalent fractions in their lowest terms:

1. $\dfrac{xy}{x^2 + xy}$.
2. $\dfrac{x^2 + xy}{x^2 - y^2}$.
3. $\dfrac{a^2 - 3a + 2}{a^2 - 5a + 6}$.
4. $\dfrac{x^2 - x - 20}{x^2 + x - 12}$.
5. $\dfrac{a^2 - 1}{4a^2 + 4a}$.
6. $\dfrac{a^2 - 2a - 3}{a^2 - 10a + 21}$.
7. $\dfrac{a^2 + 5a + 4}{a^2 + 2a - 8}$.
8. $\dfrac{x^2 + 2x - 120}{x^2 - 2x - 80}$.
9. $\dfrac{x^2 - 9x + 20}{x^2 - 7x + 12}$.
10. $\dfrac{(2a + b)^2}{4a^3 - ab^2}$.

Lesson No. 29. Review

EXERCISES

1. Multiply $3x^4 - x^2 - 1$ by $2x^4 - 3x^2 + 7$.
2. Resolve into factors:
   (1) $a^2 - 14a - 72$.
   (2) $3a^2 - 20a - 7$.
   (3) $8x^2 - 18y^2$.
   (4) $a^2 + 2ab + b^2$.

3. Reduce to its lowest terms $\frac{x^2+3x+2}{2x^2+3x+1}$.

4. Solve
$$8(9-2x)-17(25-3x)=-3.$$

5. What value of x will make the expression $(7-6x)(3-2x)$ equal to the expression $(4x-3)(3x-2)$?

6. From a rod a inches long I cut off $b-c$ inches. How many inches are left?

7. Subtract $6x^2+6$ from x^3+11x and divide the result by $x-2$.

8. What value of a will make the product of $3-8a$ and $2a$ equal to $4a(3-4a)-24$?

9. Add the square of $x+y$ to the square of $x-y$.

10. Divide the sum of $(a+b)(a+x)$ and $(a-b)(x-a)$ by $x+b$.

11. From $5(2b^2+a^2)$ take $3(4ab+a^2)$, and from $3a(a-2b)$ take $2b(3a-5b)$, and subtract the first result from the second.

12. Resolve into factors $a^2+2a+1-x^2$.

Lesson No. 30. General Review

EXERCISES

The exercises of this lesson are intended as a general review covering the fundamental rules and principles of algebra and mensuration.

1. The base and altitude of a triangle are 12 in. and 9 in. respectively. What is the altitude of a similar triangle whose base is 60 in.?

2. If the base of a rectangle is 2 ft. 6 in. and its altitude 1 ft. 6 in., what is the side of an equivalent square?

3. The base of a rectangle is four times its altitude, and its area is 21 sq. ft. 112 sq. in. Find its base and altitude.

4. Solve the following equations:

(1) $133 - 3x = x - 83$.
(2) $13 - 3x = 5x - 3$.
(3) $127 + 9x = 12x + 100$.
(4) $(x + 7)(x - 3) = (x - 5)(x - 15)$.
(5) $(x + 5)^2 - (4 - x)^2 = 21x$.

5. A and B have \$28 between them; A gives \$3 to B and then finds he has six times as much money as B. How much had each at first?

6. The price of 13 yd. of cloth is as much less than \$10 as the price of 27 yd. exceeds \$20. Find the price per yard.

7. Solve $5x + 3(7x - 4) - 2(10x - 7) = 4 - (x - 5)$.

8. If 7 be taken from a certain number and the remainder doubled it will equal the original number increased by 13. Find the number.

9. Resolve into factors:

(1) $ab - ac - b^2 + bc$.
(2) $a^2 + 4a - 21$.
(3) $a^2 + 20a + 91$.
(4) $6x^2 + x - 12$.
(5) $12a^2 - 7a + 1$.

10. Solve $12 + (3x - 7)(4x - 19) = 5(x - 2)^2 + 7(x - 3)^2$.

11. A ladder 60 ft. long stands against a vertical wall. How far must it be moved from the wall at the bottom to lower the top 5 ft.?

12. What is the length of the side of a square whose diagonal is $72\sqrt{2}$?

13. A room is 15 by 20 ft. and 12 ft. high. What is the distance from an upper corner to the opposite lower corner?

14. A rectangle is 160 ft. long and 30 ft. wide. If the length is made 40 ft. less, how much must the breadth be increased in order that the area may remain the same?

15. If the perimeter of a rectangle is 96 ft. and the length is three times the breadth, find the area.

16. If the area of a rectangle is 256 sq. ft. and the length is four times the breadth, find the length and breadth.

17. A board 16 ft. 6 in. long is 19 in. wide at one end and 11 in. wide at the other end. Find the area.

18. Divide $(x^2+2x)^2-6(x^2+2x)-27$ by x^2+2x+3.

19. Divide 396 into two parts so that one part exceeds the other by 122.

20. Divide $a^4-4a^2x^2+4ax-1$ by $a^2-2ax+1$.

21. Resolve into factors:

(1) $(a-y)^2-b^2$.
(2) $1-a^2-b^2+2ab$.
(3) $(a+1)^2-(b-1)^2$.
(4) $9a^2-12ab+4b^2-16x^2-8xy-y^2$.
(5) $a^2-4ab+4b^2-9a^2c^2$.

22. A having three times as much money as B gave B \$2; he then had twice as much as B. How much had each at first?

23. Solve the equation:

$$3(x-4)-2(11x-5)+4(5+4x)=0.$$

24. The sides forming the right angle of a right-angled triangle are 48 and 36 in. respectively. Find the length of the third side.

25. The perimeter of a square is 156 in. Find the area.

26. The area of a triangle is 437 sq. ft.; the altitude is 19 ft. Find the base.

27. Find the area of a parallelogram the base of which is 42 in. and the perpendicular height 12 in.

28. Multiply $a^2-2ab+b^2$ by $a^2+2ab+b^2$.

29. Resolve into factors:

(1) $2cd-c^2-d^2+a^2+b^2-2ab$.
(2) $a^2-6ab+9b^2-16c^2$.
(3) a^2-a-72.
(4) $a^2+a-110$.
(5) $x^2+x-132$.

30. Find the circumference of a circle the area of which is 2464 sq. in.

31. The radius of a circle is 10 in. What is the radius of a circle six times as large?

32. The driving wheel of a locomotive engine, 7 ft. in diameter, makes 96 revolutions a minute. At what rate is the train moving?

33. A wire may be bent into the form of a circle of radius 8 ft. 9 in. If the same wire were bent into the form of a square, what would be the length of its side?

34. Find by inspection the values of

(1) $(a+b)(a+2b)$.
(2) $(a-6)(a-3)$.
(3) $(ab+6)(ab+7)$.
(4) $(x-1)(x-1)$.
(5) $(3x+4)(4x-7)$.

35. The difference between the squares of two consecutive numbers is 99. What is the larger number?

36. Find the number whose double exceeds its half by 27.

37. The area of a square is 144 sq. in. Find the area of a square whose side is one-third the length of the side of the given square.

38. There are two similar right-angled triangles. The sides of the larger are twice those of the smaller. If the area of the smaller triangle is 36 sq. in., what is the area of the larger triangle?

39. A rectangle is $2\frac{1}{2}$ in. by $3\frac{1}{2}$ in. Each side is increased to eight times its present length. Find the area in square inches of the increased rectangle.

40. The area of a rectangle is 225 sq. in. Find the area of a similar rectangle whose sides and ends are one-fifth as long.

41. Find the number of square feet in the surface of a rectangular solid whose length is 9 ft., breadth 7 ft., and height 5 ft.

42. The whole surface of a cube is 486 sq. in. Find the length of each edge.

43. Resolve into factors:

(1) $36x^2 - 49y^2$.

(2) $4x^2 + 12xy + 9y^2$.

(3) $3x^2 - x - 2$.

(4) $21x^2 - 40x - 21$.

(5) $143x^2 + 8x - 16$.

44. Divide $(x^2 + x + 1)(x^3 + 1)$ by $x^2 - x + 1$.

45. Find the product of $2x^2 + 3xy + y^2$ by $2x^2 - 3xy + y^2$.

46. Solve $(3x - 17)^2 + (4x - 25)^2 - (5x - 29)^2 = 1$.

47. Solve $(x + 5)^2 - (4 - x)^2 = 21x$.

48. The base of a prism is a right triangle, the sides of which are 12 in., 16 in., and 20 in. The height is 9 ft. Find the volume in cubic feet.

49. Solve $4x^2 + 12x + 9 = 81$.

50. Find the area of the outside curved surface of a smokestack 28 in. in diameter and 24 ft. high.

NOTES, HINTS, AND ANSWERS

Lesson No. 1

Answers:

1. $(4 \times 1) + (3 \times 2) + (2 \times 3) - 4 = 12.$
2. $4 + (3 \times 4 \times 5) + 120 = 184.$
3. $(2 \times 2 \times 2) + 9 + (3 \times 3) - 25 = 1.$
4. $40 + 432 + 6 = 478.$
5. $4 + 9 + 6 + 2 - 4 = 17.$
6. $6(2 + 3 + 4) - 3 = 51.$
7. $5(2 + 3) - 2(16 - 5) = 25 - 22 = 3.$
8. $18 - 9 + 2(6 + 4) = 9 + 20 = 29.$
9. $(1 + 2)^2 + 6(15 - 3) = 3^2 + 72 = 81.$
10. $(1 + 4)2 + 12 = 10 + 12 = 22.$

Lesson No. 2

Answers:

1. $46a$. 3. $10ab$. 5. $-32bc$. 7. $14ab$.

2. $-36a$. 4. $-7a^2b$. 6. $13m$. 8. $6(a+b)$.

9. $13(x - y)$. Take $(x - y)$ just as though this quantity were represented by a single letter.

10. $8a(b + c)$.

Lesson No. 3

Answers:

1. 2. 2. 4. 3. 4. 4. 9. 5. 12.

6. 5; $3x - 18 = 7 - 2x$; $3x + 2x = 7 + 18$; $5x = 25$.

7. 3. 10. 4. 12. 8. 14. 7.

8. -2. 11. 7. 13. 9. 15. 13.

9. $x = \frac{1}{2}$.

Lesson No. 4

1. *Solution:* Let $x =$ the number.

Then
$$6x + 11 = 65;$$
$$6x = 65 - 11 = 54;$$
$$x = 9.$$

2. *Solution:* Let $x =$ the smaller number.

Then $x + 5 =$ the larger number,

and $2x + 5 =$ the sum, or 39;
$$2x = 39 - 5;$$
$$x = 17.$$

3. *Solution:* Let $x =$ the smaller part.

Then $5x - 10 =$ the other part,

and $6x - 10 =$ both parts $= 50$;
$$6x = 60;$$
$$x = 10.$$

4. *Answer:* 5 and 30.

5. *Solution:* Let $x =$ the number.

Then
$$2x + x = 72;$$
$$3x = 72;$$
$$x = 24.$$

6. *Answer:* 8.

7. *Solution:* Let $x =$ the number.

Then
$$3x = x + 46;$$
$$2x = 46;$$
$$x = 23.$$

8. *Answer:* 17 and 71.

9. *Solution:* Let $x =$ the smaller number.

Then $3x =$ the larger number,

and
$$16 - x = 30 - 3x;$$
$$3x - x = 30 - 16;$$
$$2x = 14;$$
$$x = 7.$$

10. *Answer:* A, \$83; B, \$41.

Lesson No. 5

Answers:

1. $2a+2b+2c.$
2. $7ab-13b+11.$
3. $a^2+7ab-c.$
4. $2x+5y-4z.$
5. $6ab-7c-2a^2.$
6. $-9x+15y+5z.$
7. $3x^2-5y-4xy.$
8. $10x^2+3x.$
9. $2y-22x^2-13.$
10. $x^2-6x-1.$

Lesson No. 6

1. *Solution:*

$$a+2b+(2a-3b)=a+2b+2a-3b=3a-b.$$

2. *Solution:*

$$2a-3b-(2a+2b)=2a-3b-2a-2b=-5b.$$

3. *Answer:* $2x+y.$

4. *Solution:*

$$(x-3y+2z)-(z-4y+2x)$$
$$=x-3y+2z-z+4y-2x=-x+y+z.$$

5. *Solution:*

$$2a+(b-3a)-(4a-8b)-(6b-5a)$$
$$=2a+b-3a-4a+8b-6b+5a$$
$$=2a-3a-4a+5a+b+8b-6b=3b.$$

Answers:

6. $-a+5b.$
7. $4a-6.$
8. $2a+b+c.$
9. $-x+3y.$
10. $-8x-17y.$

Lesson No. 7

1. *Answer:* $4ab-2bc.$
2. *Answer:* $b+b^2.$
3. *Answer:* $a+b.$

Solution: $3a-2b-(2b+a)-(a-5b)$
$$=3a-2b-2b-a-a+5b=a+b.$$

4. *Answer:* $c\ \ -8c^2+8c+1.$

Solution: $c^3-1-(8c^2-8c-2)=c^3-1-8c^2+8c+2$
$$=c^3-8c^2+8c+1.$$

5. *Answer:* $x = 4$.

6. *Answer:* 39 and 40.

The solution is as follows:

Let $x =$ the first number.

Then $x + 1 =$ the second number,

and $x + x + 1 = 79$;

therefore $2x = 79 - 1$, or 78,

and $x = 39$, the first number,

and $x + 1 = 39 + 1 = 40$, the second number.

7. *Answer:* $4a^2 + 3a - 19$.

8. *Answer:* $4x^3 - 7x^2 - 3x - 4$.

9. *Answer:* -9.

Solution: $3 - 12 - 0 + 0 = -9$.

10. *Answer:* The number is 8.

11. *Answer:* $x = 6$.

12. *Answer:* $x = 8$.

13. *Answer:* The parts are 80 and 100.

14. *Answer:* $x = 5$.

15. *Answer:* $2c$.

Solution: $b + c - (b - c) = b + c - b + c = 2c$.

Lesson No. 8

Answers:

1. $35x$.
2. $6b$.
3. x^5.
4. $30x^3$.
5. $42c^7$.
6. $9y^7$.
7. $12a^2b^2$.
8. $12ax^3$.
9. $12xy^3$.
10. $6a^2cd$.
11. $5a^6b^5$.
12. $84a^3b^4$.
13. $12ab^4$.
14. a^4b^4.
15. $a^3bc - a^3c^2$.
16. $6a^2b - 9b^2$.
17. $-3b^3 - 4b^2$.
18. $-3a^2c + 6b^3c$.
19. $-8b^2c + 32$.
20. $2a^5bx - 6a^4bx^2$.

Lesson No. 9

Answers:

1. $a^2 + 10a + 21$.
2. $b^2 - b - 12$.
3. $3a^2 + 2a^2b + 3ab + 2ab^2$.
4. $12a^2b + ab^2 + 6a - b^3 + 2b$.

5. $x^3 - 2x^2y + 2xy^2 - y^3$.

6. $6a^4 - 96$.

7. $a^2 - 4b^2 + 12bc - 9c^2$.

8. $3x^4 - 23x^3 - 5x^2 + 80x + 50$.

9. $a^4 - a$.

10. $2a^3 - a^2b - 4ab^2 - b^3$.

11. $a^2 - b^2$.

12. $a^4 + a^3b + a^3c - ab^2 - 3ac^2 - b^3 - b^2c - 3bc^2 - 3c^3$.

13. $x^4 - x^2 + 4x - 4$.

14. $2a^4 - 5a^3 + a^2 - 12$.

15. $a^2 - b^2 + 2bc - c^2$.

16. $a^2 + a - 6$.

17. $a^4 - b^4$.

18. $c^3 - 3c^2 - 10c + 24$.

19. $a^2 - b^2 - 2bc - c^2$.

20. $12x^2 - ax - 6a^2$.

Lesson No. 10

Answers:

1. $x^2 + 7x + 10$.
2. $x^2 + 10x + 21$.
3. $x^2 + 9x + 14$.
4. $x^2 + 13x + 36$.
5. $x^2 + 12x + 35$.
6. $x^2 + 12x + 32$.
7. $a^2 + a - 6$.
8. $a^2 - 5a + 6$.
9. $a^2 + 5a + 6$.
10. $a^2 - 2a - 15$.
11. $x^2 + 9x - 36$.
12. $x^2 + 4x - 21$.
13. $b^2 - 7b - 18$.
14. $b^2 - 7b + 12$.
15. $c^2 + c - 56$.
16. $a^2 - 1$.
17. $b^2 - 8b + 16$.
18. $a^2 - b^2$.
19. $a^2 - 2ab + b^2$.
20. $a^2 - 4b^2$.
21. $9x^2 - 1$.
22. $12x^2 + 2x - 4$.
23. $6x^2 - 5x - 6$.
24. $a^2 - 16c^2$.
25. $a^2 + 10a - 200$.
26. $x^2 - 2xy + y^2$.
27. $a^2b^2 - 4ab + 3$.
28. $a^2c^2 + 5ac + 6$.
29. $25x^2 - 15x + 2$.
30. $a^2b^2 - 9abc + 20c^2$.

Lesson No. 11

Answers:

1. 4.
2. 3.
3. 15.
4. 5.
5. 5.
6. 6.
7. 3.
8. 9.
9. 5.
10. $x = -1\frac{1}{2}$.
11. Smaller number 8; larger number 12.
12. 35 and 65.
13. A, \$18; B, \$6.
14. C = \$100; B = \$500; A = \$400.
15. \$500 and \$2500.
16. 12 years and 36 years.
17. 24 and 36.
18. 80 and 42.
19. 15 and 16.
20. 6. Let x = the number; then $2x$ exceeds one-half x by 9 or $2x - \frac{1}{2}x = 9$.

Lesson No. 12

Answers:

1. ab.
2. $3a$.
3. $2b$.
4. 5.
5. $3a$.
6. $3ab$.
7. $7b$.
8. abc.
9. $4bc$.
10. $2bc$.
11. $-3a$.
12. $-2ab$.
13. ab^2.
14. $-3c$.
15. $-3ab$.
16. $-4b$.
17. $-2ab$.
18. $6bc$.
19. $3ab$.
20. a.
21. $9a^2 + 3b$.
22. $14a + 2$.
23. $7a - b$.
24. $2ab - 3$.
25. $2a^2 - a$ or $a(2a - 1)$.
26. $x - y$.
27. $12 - ab$.
28. $4a - 1$.
29. $3b^3 + 3$.
30. $3x - 2$.

Lesson No. 13

Answers:

1. $a^3 + a^2 + a + 1$.
2. $4a^3 - 3a^2 - 5a + 6$.
3. $8a - 6b + 2ab$.
4. $a^2bc + 2b^2c^3 - 3c$.
5. $ax - 2xy$.
6. $1 - 2x^2 + 3x^4 - 4x$.
7. $a + b$.
8. $a + b + 1$.
9. $x^2z^2 + 3 - xz$.
10. $(a + b) + 3a(a + b)$.

Lesson No. 14

Answers:

1. $a+3$.
2. $x+3$.
3. $a+1$.
4. $x+3$.
5. $a+9$.
6. $x-2a$.
7. $m+13$.
8. $x-8$.
9. $3a-2$.
10. $2a+7b$.
11. $x+6$.
12. $ab+4$.
13. x^3+12.
14. $4x^2-7$.
15. $3a-4b$.

Lesson No. 15

Answers:

1. $-\frac{5}{11}$.
2. $3a^3+5a^2+2a$.
3. $2ab-a^2-2ac-b^2+2bc-c^2$.
4. $1+b-2b^2$.
5. $4a^2+3a-19$.
6. -33.
7. $4a+3x$.
8. $2y$.
9. c.
10. x^2+3x+2.

Lesson No. 16

Answers:

1. $a(a+b)$.
2. $a^2(a+2b)$.
3. $4a(m+2)$.
4. $3x(x-2y)$.
5. $bc(5a-12)$.
6. $3abc(3abc+1)$.
7. $a(4-a)$.
8. $3(1+2x^2)$.
9. $12(2a-bc)$.
10. $12a^2(ab+2)$.
11. $x(x^2+1)$.
12. $2bc(2b+c-4)$.
13. $3m(a-2+3a^2m)$.
14. $7(6a^2-2a+3)$.
15. $(a^2-5)(a+b)$.
16. $(a+7)(a-b)$.
17. $4a(b+2)(x+1)$.
18. $3a^2(1+2a)(b+c)$.
19. $4ab(1-2a)(x+y)$.
20. $x(1-5x)(x+1)$.

Lesson No. 17

Answers:

1. $(x+y)(a+b)$.
2. $(x^2+a)(x+1)$.

$$\begin{aligned} x^3+x^2+ax+a &= x^2(x+1)+a(x+1) \\ &= (x^2+a)(x+1). \end{aligned}$$

3. $(3+y)(3+x)$.

4. $(x+z)(x+y)$.

5. $(a+b)(a+2)$.

6. $(2+x)(a+x)$.

7. $(m-n)(a-b)$.

$$am-bm-an+bn=(am-bm)-(an-bn)$$
$$=m(a-b)-n(a-b)=(m-n)(a-b).$$

8. $(a-b)(x-2y)$.

9. $(a+b)(1+c^2)$.

10. $(a+c)(a+4)$.

11. $(x+y)(2m+n)$.

12. $(a-b)(c^2-2)$.

$$ac^2-2a-bc^2+2b=(ac^2-2a)-(bc^2-2b)$$
$$=a(c^2-2)-b(c^2-2)=(a-b)(c^2-2).$$

13. $(xy-d)(a+bc)$.

14. $(6a+4c)(2a-3b)$.

15. $(ax+y)(bx-y)$.

Lesson No. 18

Answers:

1. $(x+2)(x+3)$.
2. $(x+3)(x+3)$.
3. $(x+6)(x+1)$.
4. $(x+4)(x+4)$.
5. $(x+2)(x+2)$.
6. $(x+3)(x+6)$.
7. $(x+5)(x+2)$.
8. $(x+5)(x+4)$.
9. $(x+5)(x+5)$.
10. $(x+7)(x+3)$.
11. $(a+4)(a+3)$.
12. $(a+8)(a+7)$.
13. $(a+6)(a+4)$.
14. $(a+8)(a+4)$.
15. $(bc+2)(bc+1)$.
16. $(ab+5)(ab+2)$.
17. $(a^2+5)(a^2+5)$.
18. $(a^3+4)(a^3+3)$.
19. $(b+1)(b+1)$.
20. $(a+3)(a+3)$.

Lesson No. 19

Answers:

1. $(x+7)(x-2)$.
2. $(x-7)(x+2)$.
3. $(x-7)(x+1)$.
4. $(m-9)(m-1)$.
5. $(b-2)(b-1)$.
6. $(x-4)(x-3)$.
7. $(n+2)(n-1)$.
8. $(x-9)(x+2)$.
9. $(x+6)(x-2)$.
10. $(a-5)(a-2)$.
11. $(a-8)(a+3)$.
12. $(a+9)(a-4)$.

13. $(x-20)(x+10)$.
14. $(a-10)(a-4)$.
15. $(m+6)(m+4)$.
16. $(x-8)(x-6)$.
17. $(a+21)(a-20)$.
18. $(x+17)(x-15)$.
19. $(x-6)(x-3)$.
20. $(ab-5)(ab+4)$.

Lesson No. 20

Answers:

1. $(x-5)(x-2)$.
2. $(x-19)(x-10)$.
3. $(x-12)(x-11)$.
4. $(a-20)(a-10)$.
5. $(a-23)(a-20)$.
6. $(x-56)(x-1)$.
7. $(b-4)(b-3)$.
8. $(x-5)(x-4)$.
9. $(x+12)(x-5)$.
10. $(x+15)(x-3)$.
11. $(a+12)(a-1)$.
12. $(a+20)(a-7)$.
13. $(b+25)(b-12)$.
14. $(m+4)(m-1)$.
15. $(n+30)(n-13)$.
16. $(x+6)(x-11)$.
17. $(x-9)(x+2)$.
18. $(m-12)(m+3)$.
19. $(n-15)(n+4)$.
20. $(a-14)(a+1)$.
21. $(b-20)(b+5)$.
22. $(x-12)(x-3)$.
23. $(x+9)(x-5)$.
24. $(ab-18)(ab+2)$.
25. $(a+10)(a-9)$.
26. $(a+6)(a+6)$.
27. $(a+7)(a-1)$.
28. $(a+4)(a-3)$.
29. $(a-13)(a+9)$.
30. $(x+12)(x-11)$.

Lesson No. 21

Answers:

1. $28y+26z-22x$.
2. x^2+4x+5.
3. $x=8$.
4. $x=4$.
5. $-a^2$.
6. $x=6$.
7. -1
8. x^2-6x+8.
9. $x=3$.
10. $a^3-b^2-8c^3-6abc$.
11. $(a^2+5)(a^2+4)$.
12. $(x-2)(x+a)$.
13. $(x-a)(x-3)$.
14. $(x-12)(x+9)$.
15. $x=8$.

Solution: Let $x =$ the number required.

Then $x^2 + 17 = (x+1)^2$,

and $x^2 + 17 = x^2 + 2x + 1$,

and $17 = 2x + 1$,

and $2x = 16$,

and $x = 8$.

Lesson No. 22

Answers:

1. $x = 5$.	6. $x = 3$.	11. $x = 6$.	16. $x = 3$.
2. $x = 4$.	7. $x = 4$.	12. $x = 1$.	17. $x = 6$.
3. $x = 8$.	8. $x = 4$.	13. $x = 5$.	18. $x = 9$.
4. $x = 8$.	9. $x = 5$.	14. $x = 4$.	19. $x = 6$.
5. $x = 7$.	10. $x = 8$.	15. $x = 7$.	20. $x = 12$.

Lesson No. 23

Answers:

1. $x = 4$.	4. $x = 4$.	7. $x = 11$.	9. $x = 15$.
2. $x = 4$.	5. $x = 4$.	8. $x = 1$.	10. $x = 10$.
3. $x = 4$.	6. $x = 13$.		

Lesson No. 24

Answers:

1. $14x + 2$.
2. $2z - 6y$.
3. $2b + 2c - 15$.
4. $x^4 + x^2y^2 + y^4$.
5. $x^{16} - a^{16}$.
6. $x^2 + x + 1$.
7. $x^2 + xy + y^2$.
8. (1) $x(x^3 - ax^2 + bx + c)$.
   (2) $7(7y^2 - 2y + 1)$.
   (3) $(m + a)(b + x)$.
   (4) $(m + 17)(m + 6)$.
   (5) $(x - 19)(x - 10)$.
9. The value is 0.
10. $x = 4$.
11. (1) $x = 3$.
    (2) $x = 2$.
    (3) $x = 7$.
    (4) $x = 2$.
    (5) $x = 3$.
12. 70.
13. 43.
14. 23.
15. 7 and 21.

Lesson No. 25

Answers:

1. $(3x+4)(2x+3)$.
2. $(3x+3)(2x+3)$.
3. $(4x+2)(2x+1)$.
4. $(5x+4)(3x+3)$.
5. $(7x+3)(2x+5)$.
6. $(4x+7)(3x+2)$.
7. $(5x+5)(5x+3)$.
8. $(7x+7)(4x+4)$.
9. $(3x+7)(2x+8)$.
10. $(5x+9)(3x+7)$.
11. $(4x+8)(2x+7)$.
12. $(9x+3)(8x+7)$.
13. $(7x+1)(x+2)$.
14. $(4x+6)(4x+5)$; or, $2(2x+3)(4x+5)$.
15. $(6x+3)(2x+4)$; or, $2(6x+3)(x+2)$.
16. $(9x+10)(7x+2)$.
17. $(8x+1)(3x+8)$.
18. $(4x+5)(x+1)$.
19. $(7x+2)(3x+10)$.
20. $(8x+1)(2x+8)$; or $2(8x+1)(x+4)$.

Lesson No. 26

1. $-2b+c+3$.
2. $2a+3c$.
3. $2x^2+2y^2$.
4. $a-6$.
5. $11x$.
6. $x=3$.
7. $x=4$.
8. $x=-1$.
9. $x=7$.
10. $x=5$.
11. x^2-y^2.
12. $8a^2-2ab-15b^2$.
13. $6a^3+11a^2b+ab^2-3b^3$.
14. $a^4+4a^2x^2+16x^4$.
15. $16x^4-b^4$.
16. x^2+2x+4.
17. a^2-ab+b^2.
18. $5x-3$.
19. $x-2$.
20. x^2-5x+6.
21. $x=5$.
22. $x=7$.
23. $x=3$.
24. $x=3$.
25. $a=10$.

Lesson No. 27

Answers:

1. $(m+n)(m-n)$.
2. $(a+3)(a-3)$.
3. $(x+2)(x-2)$.
4. $(b+c)(b-c)$.
5. $(2x+5)(2x-5)$.
6. $(6a+7b)(6a-7b)$.
7. $(a^2+b)(a^2-b)$.
8. $(m+2)(m-2)(m^2+4)$.
9. $(2a+3b)(2a-3b)$.
10. $(x+1)(x-1)$.
11. $(a-b+d)(a-b-d)$.
12. $(x+m+n)(x-m-n)$.
13. $(a+b+c)(a+b-c)$.
14. $(x-y+m)(x-y-m)$.
15. $(a-b+4)(a-b-4)$.
16. $(a-b+7)(a-b-7)$.
17. $(6x-6)(4x+2)$; or $12(x-1)(2x+1)$.
18. $(c+a+b)(c-a-b)$.
19. $(1+x-y)(1-x+y)$.
20. $(7+x+1)(7-x-1)$; that is, $(8+x)(6-x)$.

Lesson No. 28

I

Answers:

1. a.
2. $a-3$.
3. $(a-b)^2$.
4. $a+1$.
5. $a+y$.
6. $3(a+1)$.
7. $a+4$.
8. $x-4$.
9. $x-2$.
10. $a+3$.
11. $x+4$.
12. $x-7$.
13. $x-3$.
14. $x+2$.
15. $a+4$.

II

1. $\dfrac{x}{x+y}$.
2. $\dfrac{x}{x-y}$.
3. $\dfrac{a-1}{a-3}$.
4. $\dfrac{x-5}{x-3}$.
5. $\dfrac{a-1}{4a}$.
6. $\dfrac{a+1}{a-7}$.
7. $\dfrac{a+1}{a-2}$.
8. $\dfrac{x+12}{x+8}$.
9. $\dfrac{x-5}{x-3}$.
10. $\dfrac{2a+b}{a(2a-b)}$.

Lesson No. 29

1. $6x^8 - 11x^6 + 22x^4 - 4x^2 - 7$.

2. (1) $(a-18)(a+4)$. (3) $(4x-6y)(2x+3y)$.
   (2) $(a-7)(3a+1)$. (4) $(a+b)(a+b)$.

3. $\dfrac{x+2}{2x+1}$.

4. $x = 10$.

5. $x = 1$.

6. $(a - b + c)$ in.

7. $x^2 - 4x + 3$.

8. $a = 4$.

9. $2(x^2 + y^2)$.

10. $2a$.

11. a^2.

12. $(a+1+x)(a+1-x)$.

Lesson No. 30

Answers:

1. 45 in.

2. 23.237+ sq. in.; area = 540 sq. in. *Ans.* $= \sqrt{540}$.

3. Altitude = 28 in.; base = 112 in.

Solution: 21 sq. ft. 112 sq. in. = 3136 sq. in. $3136 \div 4 = 784$ sq. in. = area of one of four squares into which rectangle can be formed.

Square root of 784 = 28 = altitude.

4. (1) $x = 54$. (2) $x = 2$. (3) $x = 9$. (4) $x = 4$. (5) $x = 3$.

5. A had \$27, and B had \$1.

Solution: Let $x =$ A's money.
Then $28 - x =$ B's money.

A gives \$3 to B, they then have $x - 3$ and $28 - x + 3$; but six times B's = A's, therefore

$$6(28 - x + 3) = x - 3;$$
$$186 - 6x = x - 3;$$
$$189 = 7x;$$
$$x = 27.$$

6. 75¢ per yard.

7. $x = 1$.

8. $x = 27$.

Solution: Let $x =$ the number.

Then
$$2(x-7) = x+13;$$
$$2x-14 = x+13;$$
$$2x - x = 13+14.$$
$$x = 27.$$

9. (1) $(a-b)(b-c)$.
(2) $(a+7)(a-3)$.
(3) $(a+13)(a+7)$.
(4) $(2x+3)(3x-4)$.
(5) $(4a-1)(3a-1)$.

10. $x = 2\frac{16}{23}$.

11. 23.979 ft.

Solution: Let $x =$ the distance.

Then
$$x^2 + 55^2 = 60^2,$$
and
$$x^2 = 3600 - 3025 = 575,$$
and
$$x = \text{the square root of } 575 = 23.979.$$

12. 72 in. The square of $72\sqrt{2}$ equals the sum of the squares of two of the sides, equals $(72 \times 72 \times 2)$. One-half of this, or (72×72), equals the square of one side; therefore one side must be 72 in.

13. 27.73+ ft. The diagonal of the floor is 25 ft. We therefore must find the hypotenuse of a right triangle whose sides are 25 ft. and 12 ft.

14. 10 ft.

15. Length = 36 ft., and breadth = 12 ft.

16. Length is 32 ft., and breadth 8 ft.

17. $20\frac{5}{8}$ ft. Find the average breadth.

18. The quotient is $x^2 + 2x - 9$.

19. Smaller, 137; larger, 259.

20. The quotient is $a^2 + 2ax - 1$.

21. The factors are as follows:

(1) $(a-y-b)(a-y+b)$.
(2) $(1-a+b)(1+a-b)$.
(3) $(a-b+2)(a+b)$.
(4) $(3a-2b-4x+y)(3a-2b+4x-y)$.
(5) $(a-2b-3ac)(a-2b+3ac)$.

22. B had \$6, and A \$18.

23. $x = -6$.

24. 60 in. The square root of the sum of the squares of the given sides equals the third side.

25. The area is 1521 sq. in.

26. 46 ft.

27. 504 sq. in. $42 \times 12 = 504$.

28. $a^4 - 2a^2b^2 + b^4$.

29. The factors are as follows:

(1) $(a - b + c + d)(a - b - c - d)$.
(2) $(a - 3b + 4c)(a - 3b - 4c)$.
(3) $(a - 9)(a + 8)$.
(4) $(a + 11)(a - 10)$.
(5) $(x + 12)(x - 11)$.

30. 176 in. 31. 24.486 sq. in. 32. 24 mi. an hour.

33. 165 in. Circumference of circle or perimeter of square equals 660 in.

34. The products are as follows:

(1) $a^2 + 3ab + 2b^2$.
(2) $a^2 - 9a + 18$.
(3) $a^2b^2 + 13ab + 42$.
(4) $x^2 - 2x + 1$.
(5) $12x^2 - 5x - 28$.

35. The larger number is 50.

Solution:

$$(x + 1)^2 - x^2 = 99;$$
$$2x + 1 = 99;$$
$$x = 49 = \text{smaller number.}$$

36. $x = 18$.

37. 16 sq. in.

38. The area of the larger triangle is 144 sq. in.

39. 560 sq. in. $(2\frac{1}{2} \times 8)(5\frac{1}{2} \times 8) = 560$.

40. 9 sq. in.

41. 286 sq. ft.

42. 9 in. $486 \div 6 = 81$. $\sqrt{81} = 9$.

43. The factors are as follows:

(1) $(6x - 7y)(6x + 7y)$.

(2) $(2x + 3y)(2x + 3y)$.

(3) $(x - 1)(3x + 2)$.

(4) $(7x + 3)(3x - 7)$.

(5) $(11x - 4)(13x + 4)$.

44. The quotient is $x^3 + 2x^2 + 2x + 1$.

45. The product is $4x^4 - 5x^2y^2 + y^4$.

46. $x = 6$. **47.** $x = 3$. **48.** 6 cu. ft. **49.** $x = 3$.

50. 176 sq. ft. $28 \times 3\frac{1}{7} = 88$ in. $88 \div 12$ and multiplied by $24 = 176$.

V

A FIRST COURSE IN GEOMETRY

A FIRST COURSE IN GEOMETRY

The elementary course in geometry to be presented here is based upon the English system of teaching the subject. The first book of Euclid's famous and time-honored treatise is given complete, and it is hoped that it is presented in such a way as to make the subject intelligible to students studying at home. The following directions should be specially noted:

1. Nothing is to be committed to memory.

2. In learning a proposition, follow the construction and proof slowly and intelligently, making sure that you understand every move and statement.

3. When you know a proposition you should write it out complete, following the construction and reasoning step by step, but without reference to the text. You should not leave one proposition for the next until you can do this.

4. The questions and exercises are supplementary, and are intended as a test of your knowledge of the propositions previously learned.

5. The definitions required are given in connection with the propositions as they are needed, rather than at the beginning of the course, as is usual in most text-books.

Lesson No. 1

DEFINITIONS

1. A **point** is that which has position, but which has no size.

2. A **line** is length without breadth.

3. A **straight line** is that which lies evenly between its extreme points. It is sometimes defined as the shortest

distance between two points. A **finite** straight line is an ended straight line.

4. A **triangle** is a figure contained by three straight lines. If the lines are of equal length the triangle is said to be **equilateral.**

5. A **circle** is a plane (flat) figure contained by one line called the **circumference,** and is such that all straight lines drawn from a certain point within the figure to the circumference are equal. This point is called the **center** of the circle.

6. The **radius** of a circle is any straight line drawn from the center to the circumference. All radii of a circle are equal.

AXIOM

1. Things which are equal to the same thing are equal to one another. (**Axioms** are self-evident truths, therefore requiring no proof.)

Proposition 1. Problem

To describe an equilateral triangle upon a given finite straight line.

Note I. — This is the proposition or general statement of the thing required to be done. In this instance what is required to be done is to solve a problem — to construct something. We have to make an equilateral triangle, and after we have it made we must prove that it really is equilateral. We first draw the straight line AB, and then proceed, step by step, with the construction and proof.

Note II. — The propositions in Euclid are of two kinds: (1) Problems, things required to be constructed. (2) Theorems, things required to be proven.

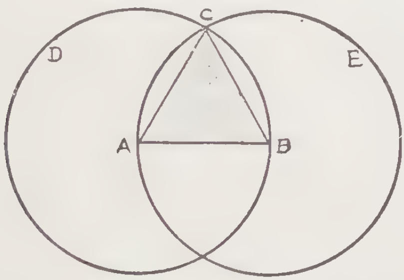
Let AB be the given straight line.

It is required to describe an equilateral triangle on AB.

Construction: With center A and radius AB, describe the circle BCD. With center B and radius AB, describe the

circle ACE. From the point C, where the circles cut, draw CA and CB. Then ACB will be the required equilateral triangle.

Proof: Since A is the center of the circle BCD, therefore AC is equal to AB, and since B is the center of the circle



ACE, therefore CB is equal to AB. But by Axiom 1, things which are equal to the same thing are equal to one another; therefore, AC is equal to CB, and AB, AC, and CB are all equal.

Wherefore, the triangle ACB described on AB is equilateral.

QUESTIONS AND EXERCISES

1. How many dimensions has a straight line? (Length only.)
2. How many dimensions has a shadow? A board?
3. Is a circumference a plane figure? Is a circle?
4. How many straight lines can be drawn between four given points? (Six.)
5. If F be the other point of intersection of the circles, prove that AF is equal to CB.
6. Produce AB both ways, making a line three times as long.
7. Make an equilateral triangle, each side of which is $2\frac{1}{2}$ in.

Lesson No. 2

The beginner should notice carefully the different parts of a proposition. The general statement printed in italics at the beginning of the proposition is called the **enunciation**. The condition subject to which any problem is to be solved or theorem proved is called the **hypothesis**. For example, the hypothesis of Proposition 1 is that the equilateral triangle must be constructed upon a given finite straight line; that is, on a straight line whose position and length have been fixed beforehand. Then follows the particular directions for the construction of the particular figure, and then the proof that it really is such as was required. The student must construct the figure step by step according to the directions.

AXIOMS

2. If equals be added to equals, the wholes are equal.
3. If equals be taken from equals, the remainders are equal.

PROPOSITION 2. PROBLEM

From a given point to draw a straight line equal to a given straight line.

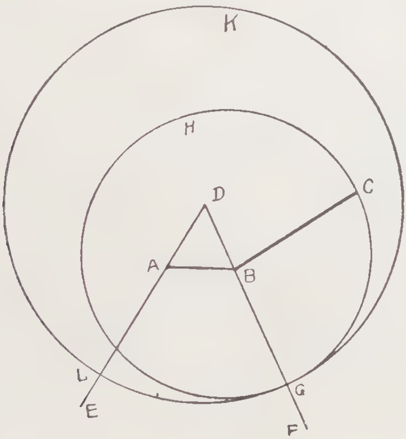
Let A be the given point and BC the given straight line.

It is required to draw from A a straight line equal to BC.

Construction: Join AB. On AB describe an equilateral triangle DAB. (This we have shown that we are able to do by Proposition 1.) Produce DA and DB to any distant points E and F. With center B and radius BC describe the circle CGH, cutting DF at G. With center D and radius DG describe the circle GLK, cutting DE in L. Then shall the line AL drawn from the point A equal BC.

Proof: Since B is the center of the circle CGH, therefore BC is equal to BG, and since D is the center of the circle GLK, therefore DL is equal to DG. But the part DA of DL is equal to the part DB of DG, therefore by Axiom 3 the remainder AL is equal to the remainder BG. Hence

AL and BC are each equal to BG, and by Axiom 1 AL is equal to BC.



Wherefore the required line AL is drawn from the point A.

QUESTIONS AND EXERCISES

1. This proposition should be practiced with varied positions of the given line and point. For example, draw the figure necessary for the construction of the problem when the point A is in the line BC.

2. Draw the figure when the point A coincides with the end B of the line BC.

3. What is the condition under which the point D would lie on the circumference of the smaller circle?

4. What definition is made use of in the proof of this proposition?

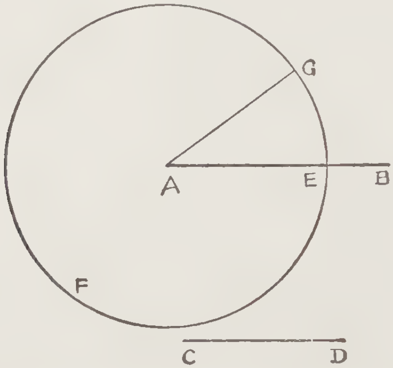
Proposition 3. Problem

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and CD be the given straight lines, AB being the greater.

It is required to cut off from AB a part equal to CD.

Construction: From A draw AG equal to CD (Proposition 2). With center A and radius AG describe a circle GEF cutting AB in E. Then shall AE equal CD.



Proof: Since AE is equal to AG and CD is equal to AG, therefore AE is equal to CD (Axiom 1).

Wherefore from AB has been cut off a part AE equal to CD.

QUESTIONS AND EXERCISES

1. Produce the smaller of two straight lines so that it shall be equal to the greater, and prove the problem.

2. Why cannot we measure CD with a pair of compasses and mark off the length on AB?

3. From AB cut off a part equal to twice CD, and prove the problem.

4. In what way is Proposition 1 made use of in Proposition 3?

Lesson No. 3

Remember that you are not to learn a proposition "by heart." You should know the enunciation (printed in italics at the beginning of the proposition), and be able to repeat it

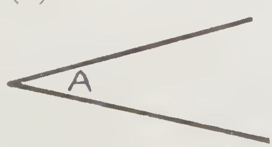
word for word, giving the correct number of the proposition, but this is all the memorizing which is necessary. Keep the enunciation before you throughout the proposition. If you lose sight of what you are trying to do, you will surely get muddled. Go slowly, step by step, over each part of the construction and proof. When you think that you understand it, try to go through the proposition without the text and with a figure of your own drawing, referring to the text only when you forget a step. Never copy the figure from the paper, but draw it, bit by bit, as you write down the steps of the construction. This caution is specially applicable to problems. Do not make your figures and letters too small. Use different letters from those in the text. Draw your figures as accurately as possible, using ruler and compasses where necessary.

Problems and Theorems

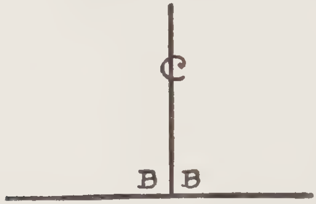
The student must understand clearly the difference, already referred to, between what are known in geometry as problems and theorems. A **problem** proposes to effect some geometrical construction; for instance, as in Proposition 1, to construct an equilateral triangle. A **theorem** proposes to demonstrate or prove some geometrical truth. Propositions 1, 2, and 3 are problems; Proposition 4 is a theorem.

DEFINITIONS

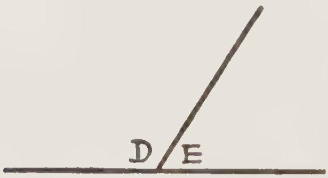
7. A **plane rectilineal angle** is the inclination of two straight lines to each other in a plane, which meet, but are not in the same straight line (A).



8. When a straight line standing on another straight line makes the adjacent angles equal, each is called a **right angle** (B). The line which stands upon the other is called a **perpendicular** to it (C).



9. An angle which is greater than a right angle is called an **obtuse angle** (D). An angle which is less than a right angle is called an **acute angle** (E).



NOTE. — The vertex of an angle is the point at which the lines which form the angle meet. One angle or line or surface is said to coincide with another angle or line or surface when it fits evenly upon it. The student must note that the length of the lines forming an angle has nothing whatever to do with the size of the angle. The size is the number of degrees which the angle contains. An angle is said to contain as many degrees as it *subtends* degrees in the circumference of a circle drawn with the vertex of the angle as center. Any complete circle has 360 degrees. Therefore a right angle will contain 90 degrees. An obtuse angle must contain more than 90 degrees, and an acute angle less than 90 degrees. The lines which meet and form an angle are said to contain that angle.

AXIOMS

4. Two straight lines cannot inclose a space.
5. Magnitudes which coincide with one another are equal.

ON TRIANGLES

Triangles have six parts, namely, three angles and three sides. Two triangles are said to be identically equal, or equal in all their parts, when one can be placed over the other so that it exactly covers it. Then all the six parts of the one are equal respectively to the six parts of the other.

Triangles may be of the same size, though different in shape. This is expressed by saying that the triangles are equal in area. That is to say, triangles may be equal in area and yet be entirely different in all their parts.

If it is known that three parts of one triangle are equal to three parts of another triangle, it can generally be proved that the remaining parts are equal.

But this is not always the case. For instance, if the three angles of one triangle are equal to the three angles of another triangle, it does not follow that the three sides are equal, for we can have two triangles, one small and the other large, which have the three angles of the one respectively equal to the three angles of the other.

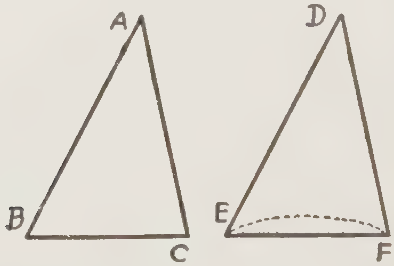
Proposition 4. Theorem

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal, then shall their bases or third sides be equal, and the triangles shall be equal in area, and their remaining angles shall be equal, each to each, namely, those to which the equal sides are opposite; that is to say, the triangles shall be equal in all respects.

Let ABC, DEF be two triangles, which have the side AB equal to the side DE, and the side AC equal to the side DF, and also the contained angle BAC equal to the contained angle EDF.

Then shall the base BC be equal to the base EF, and the triangle ABC shall be equal to the triangle DEF in area; and the remaining angles shall be equal, each to each, to which the equal sides are opposite, namely, the angle ABC to the angle DEF and the angle ACB to the angle DFE.

Proof: If the triangle ABC be applied to the triangle DEF, so that the point A may be on the point D and the straight line AB may fall along the straight line DE, then, because AB is equal to DE, therefore the point B must coincide with the point E. And because AB falls along DE and the angle BAC is equal to the angle EDF, therefore AC must fall along DF. And because AC is equal to DF, therefore the point C must coincide with the point F. Then B, coinciding with E, and C with F, the base BC must



coincide with the base EF; for, if not, two straight lines would inclose a space, which is impossible (Axiom 4). Thus the base BC coincides with the base EF, and is, therefore, equal to it (Axiom 5). And the triangle ABC coincides with the triangle DEF, and is, therefore, equal to it in area (Axiom 5). And the remaining angles of the one coincide with the remaining angles of the other, and are, therefore, equal to them, namely, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

That is, the triangles are equal in all respects.

QUESTIONS AND EXERCISES

NOTE.—Proposition 4 is of great importance, and the beginner should not attempt to proceed further until he has thoroughly mastered it.

1. If two triangles have two sides of the one equal to two sides of the other, must the triangles be equal in all respects?

2. AB is a straight line; D is its middle point; DC is a line at right angles to AB. Prove that CA is equal to CB.

3. Prove that the diagonals of a square are equal. Each diagonal forms one side of two triangles. Begin by comparing these two triangles.

4. Prove that the four straight lines joining the middle points of the sides of a square are equal. (Proceed, as in the last exercise, by comparing triangles.)

Lesson No. 4

CLASSIFICATION OF TRIANGLES

According to their sides, triangles are divided into three classes:

1. Equilateral. 2. Isosceles. 3. Scalene.

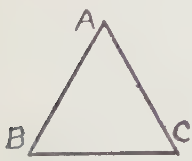


FIG. 1.

FIG. 2.

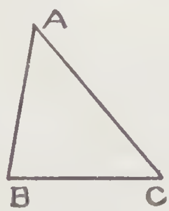


FIG. 3.

An **equilateral triangle** is one that has three equal sides. Thus, if AB, BC, CA are all equal, the triangle ABC is equilateral.

An **isosceles triangle** is one that has two equal sides. Thus, if AB is equal to AC the triangle ABC is isosceles.

A **scalene triangle** is one that has three unequal sides. Thus, if AB, BC, CA are all unequal, the triangle ABC is scalene.

According to their angles, triangles are divided also into three classes:

1. Right-angled. 2. Obtuse-angled. 3. Acute-angled.

Fig. 1.

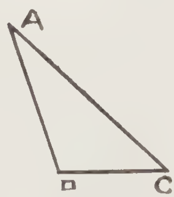


Fig. 2.

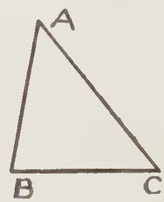


Fig. 3.

A **right-angled triangle** is one that has a right angle. Thus, if ABC is a right angle, the triangle ABC is right-angled. A triangle cannot have more than one right angle.

An **obtuse-angled triangle** is one that has an obtuse angle. Thus, if ABC is an obtuse angle, the triangle ABC is obtuse-angled. A triangle cannot have more than one obtuse angle.

An **acute-angled triangle** is one that has three acute angles. Thus, if each of the angles of the triangle ABC is acute, the triangle is acute-angled.

Any side of a triangle may be called the base. In an isosceles triangle the side which is neither of the equal sides is usually called the base. In a right-angled triangle one of the sides containing the right angle is usually called the base, and the other the perpendicular. The side opposite the right angle is called the hypotenuse. Any of the angular points of a triangle may be called a vertex. If one of the sides of a triangle has been called the base, the

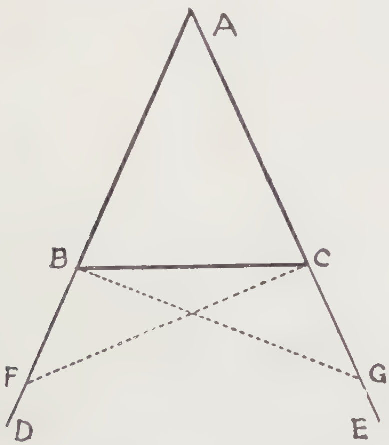
angular point opposite that side is usually called the vertex.

Proposition 5. Theorem

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base shall also be equal to one another.

Let ABC be an isosceles triangle, having the side AB equal to the side AC, and let the straight lines AB and AC be produced to D and E.

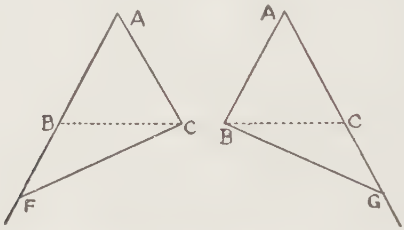
Then shall the angle ABC be equal to the angle ACB, and the angle CBD equal to the angle BCE.



Construction: In BD take any point F, and from AE the greater cut off AG equal to AF the less (Proposition 3). Join FC and GB.

Proof: Then, in the triangles FAC and GAB, FA equals GA, and AC equals AB, and the contained angle at A is common to both triangles; therefore, the triangle FAC is equal to the triangle GAB in all respects (Proposition 4). Again, because AF is equal to AG, and the parts AB and AC are equal, therefore the remainder BF is equal to the

remainder CG. Then, in the two triangles BFC and CGB, BF is equal to CG, and FC has been shown equal to GB, and the angle BFC has also been shown to be equal to the angle CGB; therefore, the triangles BFC and CGB are equal in all respects (Proposition 4).



Now, it has been shown that the whole angle ABG is equal to the whole angle ACF, and that the parts CBG and BCF are equal, therefore the remaining angle ABC is equal to the remaining angle ACB, and these are the angles at the base of the triangle ABC. Also, it has been shown that the angle FBC is equal to the angle GCB, and these are the angles at the other side of the base.

NOTE.—This Proposition 5 is, with perhaps the exception of Proposition 47, the most famous proposition in all geometry. To beginners it often seems difficult, and yet it is not really difficult, especially if the several parts of its proof be mastered with care. On account of its difficulty to beginners it has for ages been known as the *pons asinorum*, the "asses' bridge," because dunces were supposed to fall over it, and not to be able to proceed further. It is to be hoped that it will not prove a *pons asinorum* to any of our students.

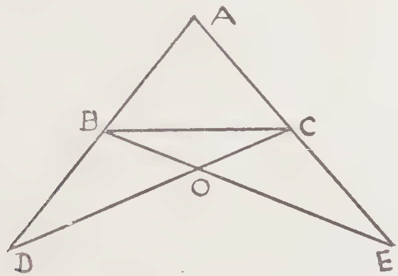
QUESTIONS AND EXERCISES

1. Draw a right-angled isosceles triangle.

2. Does the size of an angle depend upon the length of its arms?

3. What is meant by an equiangular figure? What by a rectangular figure? What by a rectilineal figure?

4. Mention the names of eight triangles in the accompanying figure:



5. One of the equal sides of an isosceles triangle is 2 ft. 3 in., and the base is 1 ft. 10 in. Find the perimeter of the triangle.

6. Can three straight lines be drawn so that even if they are produced they will not inclose a space?

7. Prove that if two angles of a triangle be unequal the sides opposite to them will also be unequal.

8. Two isosceles triangles, ABC and DBC, stand on the same base, BC, and on opposite sides of it. Prove that the angle ABD is equal to the angle ACD.

9. Two isosceles triangles, ABC and DBC, stand on the same base, BC, and on the same side of it. Prove that the angle ABD is equal to the angle ACD.

10. In Proposition 5 how far must the equal sides be produced? Must they both be produced to the same length?

Lesson No. 5

INTRODUCTORY NOTES AS TO CONVERSE PROPOSITIONS

The following statement would be accepted as true:

If rain is falling, there are clouds overhead.

It does not follow that

If there are clouds overhead, rain is falling.

The second statement is called the *converse* of the first. One proposition is said to be the converse of another when the hypothesis of the one is the conclusion of the other.

In the above illustration "rain is falling" is the hypothesis of the first proposition and the conclusion of the second; "there are clouds overhead" is the conclusion of the first and the hypothesis of the second. Therefore the second proposition is the converse of the first. But the first proposition is true, while the second is not true.

We see that a converse proposition of a proposition already proved is not necessarily true. However, it often happens that a converse proposition is true, but it must be proved before it can be accepted.

The converse of the proposition —

If two sides of a triangle be equal to one another, then the angles which are subtended by the equal sides shall be also equal to one another,

is —

If two angles of a triangle be equal to one another, then the sides which subtend the equal angles shall be also equal to one another.

We have now to find a method to prove this converse proposition true.

There are two methods by which converse propositions may frequently be proved — either by (1) the exhaustive method, or (2) a method called "reductio ad absurdum," which literally translated means "a reducing to absurdity."

The exhaustive method is not used in the first twelve propositions, so we will not discuss it. Let us try to understand the second method.

"Reductio ad absurdum" is used when we wish to prove the truth of a proposition by showing that the opposite of that proposition cannot *possibly* be true. For example:

A thing is *either* (1) a ship *or* (2) not a ship.

An article is *either* (1) white *or* (2) not white.

An animal *either* (1) has a tail *or* (2) has no tail.

Two things are *either* (1) equal *or* (2) unequal.

Now, in each case *either* (1) *or* (2) must be true.

If we *assume* (2) is true, and by reasoning from that assumption arrive at an absurdity, we conclude *either* that our reasoning has been incorrect *or* our assumption incorrect.

But if no flaw can be found in our reasoning, our assumption — namely, that (2) is true — must have been incorrect.

Therefore, supposing that (1) or (2) must be true and that we have proved that (2) cannot be true, it follows that (1) must be true.

Hence in a "reductio ad absurdum" *we assume as true the opposite of the fact that we wish to prove.*

If with that assumption we arrive at an absurdity, we conclude that the fact that we wish to prove true is true.

We have seen that if two sides of a triangle are equal the angles opposite to them are equal.

We have to prove that *if two angles of a triangle are equal the sides opposite to them are equal.*

We employ "reductio ad absurdum."

We assume as true the opposite of the fact that we wish to prove. Therefore, we assume that the sides are unequal.

If so, one side must be greater than the other.

Proposition 6. Theorem

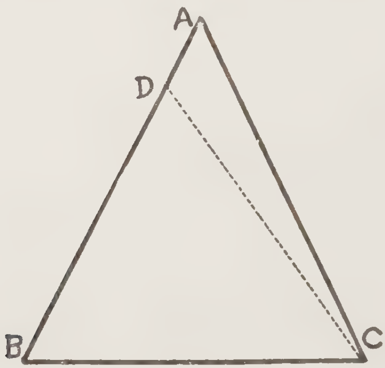
If two angles of a triangle be equal to one another, then the sides also which subtend or are opposite to the equal angles shall be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB; then shall the side AC be equal to the side AB.

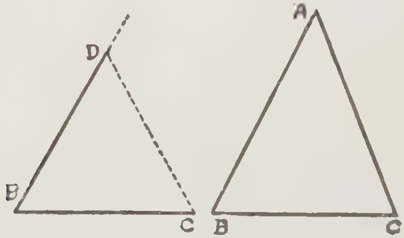
Construction: For if AC be not equal to AB, one of them must be greater than the other. If possible let AB be the

greater, and from it cut off DB equal to AC (Proposition 3) and join DC.

Proof: Then in the triangles DBC and ABC, because DB is equal to AC and BC is common to both, and the contained



angle DBC is equal to the contained angle ACB, therefore the triangle DCB is equal in area to the triangle ACB (Proposition 4); that is, the part equal to the whole, which is absurd.



Therefore, DB is not equal to AC. In a similar way it can be shown that no *other* part of AB is equal to AC, and that no part of AC is equal to AB. Therefore, AB is not unequal to AC; that is, AB is equal to AC.

QUESTIONS AND EXERCISES

1. Can a number of geometrical lines placed close to one another form a surface?

2. Would it be possible to draw a straight line upon a surface that is not plane? If so, give an example.

3. How many arms has an angle? What name is given to the point where the arms meet?

4. If two adjacent angles are equal, what name is given to the arm which is common to the two angles?

5. When the hypotenuse of a triangle is mentioned, of what sort must the triangle be?

6. If $ABCD$ is a square and the bisectors AO and BO of the angles at A and B meet in O, prove that AO is equal to BO.

7. AD and BD, the bisectors of two angles of an equilateral triangle, meet in D. Prove that AD is equal to BD.

Lesson No. 6

Proposition 7. Theorem

When two triangles on the same base and on the same side of it have their sides terminated at one end of the base equal, then the sides terminated at the other end cannot be equal.

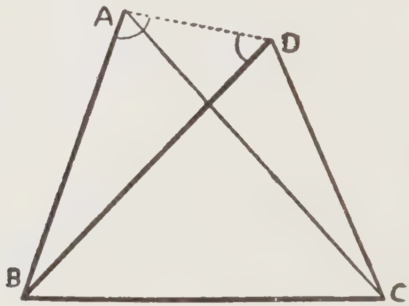
Note. — There are three possible cases of this proposition as shown by the figures: (1) when the vertex of each triangle is outside of the other, (2) when the vertex of one triangle is inside of the other triangle, (3) when the vertex of one triangle is on a side of the other.

If it be possible, let the triangles ABC and DBC, standing on the same side of BC, have the side AB equal to DB, and the side AC equal to DC.

1. Let the vertex D fall outside of the triangle ABC.

Proof: Join AD. Then, since AB is equal to DB, therefore the angle BAD is equal to the angle BDA (Proposition 5). But angle BAD is greater than the angle CAD,

therefore BDA must be greater than CAD, and CDA must be still greater than CAD. But since CD is equal to CA, therefore these angles CAD and CDA are equal (Proposi-

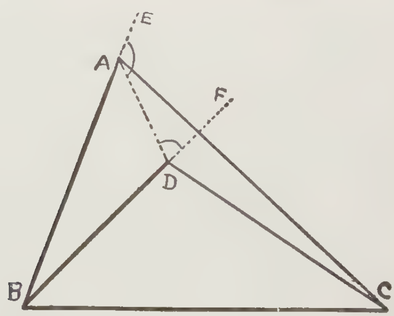


tion 5). That is, the angle CDA is both equal to and greater than CAD, which is impossible.

Therefore, if BA is equal to BD, CA *cannot* be equal to CD.

2. Let the vertex D fall within the triangle ABC.

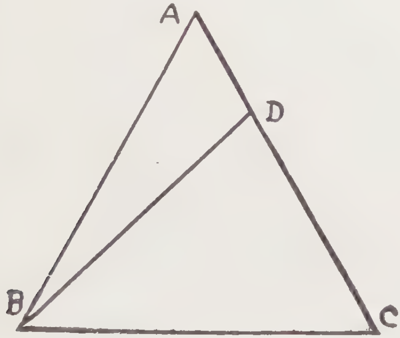
Proof: Join AD. Produce BA and BD to E and F. Then, since AB is equal to DB, therefore the angle EAD is



equal to the angle FDA (Proposition 5). But EAD is greater than CAD, therefore FDA is greater than CAD, and CDA must be still greater than CAD. But since CA

is equal to CD, therefore the angle CDA is equal to the angle CAD (Proposition 5). That is, the angle CDA is both equal to and greater than CAD, which is impossible.

Therefore, if BA is equal to BD, CA *cannot* be equal to CD.



3. Let the vertex D fall on the line AC.

This case needs no proof. It is evidently impossible for CA to be equal to CD.

NOTE. — This negative theorem is required by Euclid for only the eighth proposition. Some writers have used another proof of Proposition 8, omitting the seventh proposition altogether.

QUESTIONS AND EXERCISES

1. Show that on the same base and on the same side of it there can be but one equilateral triangle.
2. What is the method of proof called which is used in this proposition?
3. What is meant by the vertex of a triangle?

Lesson No. 7

On the Equality of Triangles

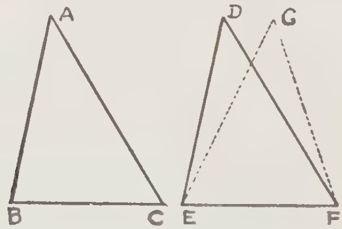
We have already shown in Proposition 4 that, if certain conditions are given, we can prove that two triangles are identically equal. The conditions given in Proposition 4

are not the *only* conditions which will enable us to prove two triangles identically equal. If two triangles have the three sides of one equal to the three sides of the other, the triangles are identically equal. The proof is effected by means of the proposition last proved (Proposition 7).

Proposition 8. Theorem

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, then the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides of the other.

Let *ABC*, *DEF* be two triangles having the two sides *BA*, *AC* equal to the two sides *ED*, *DF*, each to each, namely, *BA* to *ED* and *AC* to *DF*, and also the base *BC* equal to the base *EF*; then shall the angle *BAC* be equal to the angle *EDF*.



Proof: If the triangle *ABC* be applied to the triangle *DEF*, so that the point *B* may be on *E* and the straight line *BC* along *EF*, then because *BC* is equal to *EF*, therefore the point *C* must coincide with the point *F*. Then *BC* coinciding with *EF*, it follows that *BA* and *AC* must coincide with *ED* and *DF*; for, if not, they would have a different situation, as *EG*, *GF*. Then, on the same base and on the same side of it there would be two triangles having their *conterminous* sides equal. But this is impossible (Proposi-

tion 7). Therefore the sides BA, AC coincide with the sides ED, DF.

That is, the angle BAC coincides with the angle EDF, and is therefore equal to it.

NOTE 1. — It follows from Proposition 4 that the triangles ABC and DEF are equal in every respect.

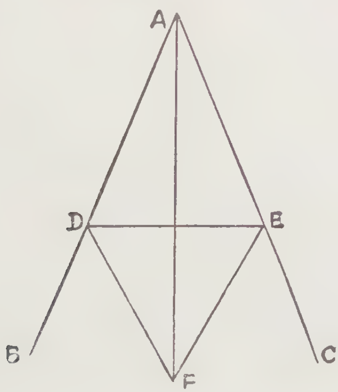
NOTE 2. — We have now two ways of proving two triangles to be equal in all respects. We must have two sides of the one equal to two sides of the other *and also* either (1) the angles contained by these equal sides equal (Proposition 4) or (2) the third sides equal (Proposition 8).

PROPOSITION 9. PROBLEM

To bisect a given angle — that is, to divide it into two equal parts.

Let BAC be the given angle; it is required to bisect it.

Construction: In AB take any point D and from AC cut off AE equal to AD (Proposition 3). Join DE, and on DE,



on the side remote from A, describe an equilateral triangle DEF (Proposition 1). Join AF. Then shall the straight line AF bisect the angle BAC.

Proof: In the two triangles DAF, EAF, DA is equal to EA, and AF is common to both: and DF is equal to EF;

therefore the angle DAF is equal to the angle EAF (Proposition 8).

Therefore the given angle BAC is bisected by the straight line AF.

QUESTIONS AND EXERCISES

1. Prove by Proposition 8 that the straight line which joins the vertex to the middle point of the base of an isosceles triangle is at right angles to the base.

2. Why in Proposition 9 is the equilateral triangle constructed on the side of DE remote from A?

3. Show how to divide a given rectilineal angle into four equal parts.

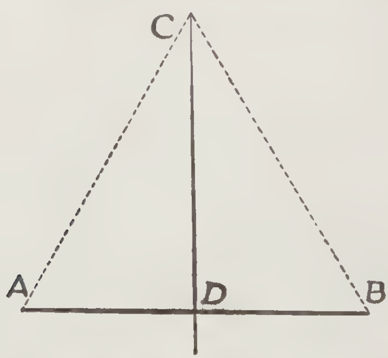
Lesson No. 8

Proposition 10. Problem

To bisect a given finite straight line.

Let AB be the given straight line; it is required to bisect it.

Construction: On AB describe an equilateral triangle ABC (Proposition 1), and bisect the angle ACB by the



straight line CD (Proposition 9), cutting AB in D. Then AB is bisected at the point D.

Proof: In the triangles ACD and BCD, because AC equals BC and CD is common to both, and the contained

angle ACD is equal to the contained angle BCD, therefore the triangles are equal in all respects, and the base AD is equal to the base DB (Proposition 4).

Therefore, AB is bisected at the point D.

QUESTIONS AND EXERCISES

1. Why in the enunciation is it necessary for it to be said that the straight line is finite?

2. Divide a given straight line into four equal parts.

3. Would not an isosceles triangle described on the given line answer as well as an equilateral? Why, then, in the construction of the problem, is an equilateral triangle used instead of an isosceles triangle?

4. Can you prove that ADC is a right angle?

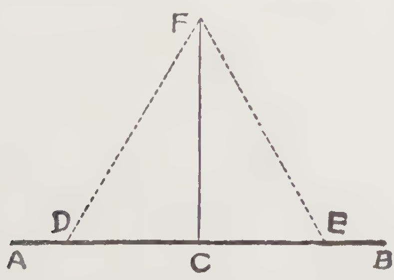
5. Show how to find a line half as long again as a given line.

6. Find a point which is equally distant from two given points.

Proposition 11. Problem

To draw a straight line at right angles to a given finite straight line from a given point in the same.

Let AB be the given straight line, and C the given point in it; it is required to draw from the point C a straight line at right angles to AB.



Construction: In AC take any point D, and from CB cut off CE equal to CD (Proposition 3). On DE describe

the equilateral triangle DFE (Proposition 1). Join CF. Then shall the straight line CF be at right angles to AB.

Proof: For in the triangles DCF and ECF, because DC is equal to EC and CF is common to both, and DF is equal to EF, therefore the angle DCF is equal to the angle ECF (Proposition 8), and these are adjacent angles. But when a straight line standing on another straight line makes the adjacent angles equal to one another, each of these angles is called a right angle (Definition 8); therefore each of the angles DCF and ECF is a right angle; therefore CF is at right angles to AB and it has been drawn from the point C.

QUESTIONS AND EXERCISES

1. How do problems differ from theorems?
2. Would an isosceles triangle answer in the construction in this proposition?
3. At a given point in a given straight line make an angle equal to half a right angle.
4. Construct an isosceles right-angled triangle.
5. Can you prove Proposition 11 without using Proposition 8?

Proposition 12. Problem

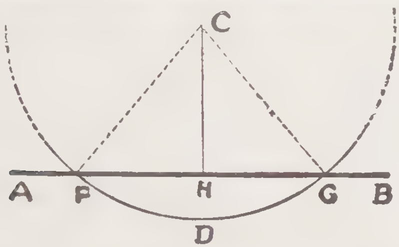
To draw a straight line perpendicular to a given straight line of unlimited length from a given point without it.

Let AB be the given straight line which, since it is of unlimited length, may be produced in either direction, and let C be the given point without it. It is required to draw from C a straight line perpendicular to AB.

Construction: On the side of AB remote from C take any point D, and from center C with radius CD describe the circle FDG, meeting AB at F and G. Bisect FG at H (Proposition 10), and join CH. Then shall the straight line CH be perpendicular to AB. Join CF and CG.

Proof: In the triangles FHC, GHC, FH is equal to GH, and HC is common to both; and CF is equal to CG, being

radii of the circle FDG; therefore the angle CHF is equal to the angle CHG (Proposition 8), and these are adjacent angles. But when a straight line standing on another straight line makes the adjacent angles equal to one another,



each of these angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it (Definitions).

Therefore CH is a perpendicular drawn to the given straight line AB from the given point C without it.

QUESTIONS AND EXERCISES

1. Why is the line given of unlimited length?

2. Why is the point D taken on the other side of AB?

3. Under what condition would CFG be an isosceles triangle?

4. What difference would it make in the proof if instead of bisecting FG in H we bisected the angle FCG by CH meeting AB in H?

Lesson No. 9

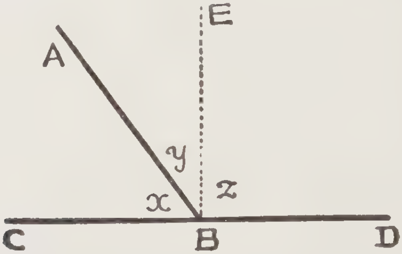
PROPOSITION 13. THEOREM

The angles which one straight line makes with another straight line, on one side of it, are either two right angles or are together equal to two right angles.

Let the straight line AB meet CD at B. Then the angles ABC and ABD shall be each right angles or they shall be together equal to two right angles.

Construction: If ABC is equal to ABD, they are two right angles (definition of a right angle).

If ABC is not equal to ABD, from B draw BE at right angles to CD (Proposition 11). Denote the three angles at B by the letters x, y, and z, as in the figure.



Proof: Now the angles x, y, and z together are equal to the two angles ABC and EBD, and the angles x, y, and z together are also equal to the two angles ABC and ABD. Therefore the two angles ABC and ABD are equal to the two angles EBC and EBD. But these two angles are right angles; therefore the two angles ABC and ABD must be equal to two right angles.

Proposition 14. Theorem.

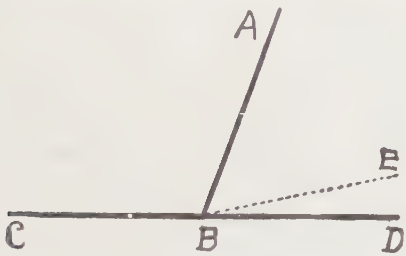
If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines must be in one and the same straight line.

At the point B in the straight line AB, let CB and BD on opposite sides of AB make the angles CBA and ABD together equal to two right angles. Then CB shall be in the same straight line with BD.

Construction: For, if not, if possible produce CB in some other direction BE.

Proof: Then, since AB meets the straight line CBE at B, therefore the angles CBA and ABE are equal to two

right angles (Proposition 13). But the angles CBA and ABD equal two right angles; therefore the angles CBA and ABE are equal to the angles CBA and ABD. From these equals take away the common angle CBA. Then the remaining angle ABE is equal to the remaining angle ABD, or the



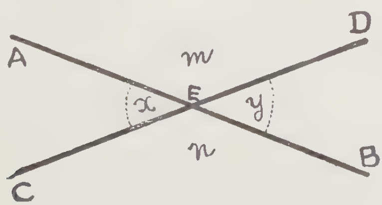
part equal to the whole, which is impossible. Therefore BE is not in the same straight line with BC. And in the same manner it may be known that no other line but BD is in the same straight line with BC.

Therefore CB is in the same straight line with BD.

Proposition 15. Theorem

If two straight lines cut one another, the opposite vertical angles are equal.

Let the straight lines AB and CD cut at E. Denote the angles at the point of intersection by the letters x, y, m,



and n. Then shall the angle x be equal to the angle y and the angle m be equal to the angle n.

Proof: Since AE meets CD in E, therefore the angles x and m equal two right angles (Proposition 13); and since DE meets AB in E, therefore the angles m and y equal two right angles. Therefore the angles x and m equal the angles m and y. Take away the common angle m and the remaining angle x is equal to the remaining angle y. In the same way it may be shown that the angle m is equal to the angle n.

QUESTIONS AND EXERCISES

1. Write out the full proof of the second part of Proposition 15.

2. If AB and CD bisect each other in E, prove that the triangles AEC and DEB are equal in every respect.

3. Of what proposition is the 14th the converse?

4. What method of proof is adopted in Proposition 14?

5. If the two angles at the base of a triangle are equal, prove that the angles at the other side of the base are equal.

6. Make an angle equal to half a right angle.

7. What is the converse of Proposition 15?

Lesson No. 10

Proposition 16. Theorem

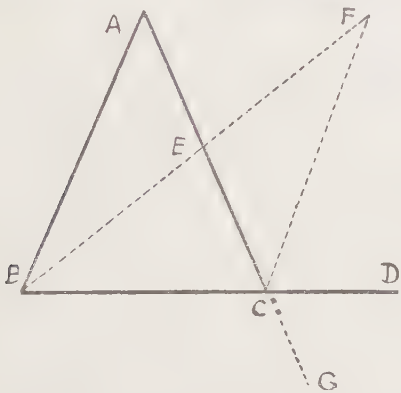
If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

Let ABC be a triangle, and let one side BC be produced to D; the exterior angle ACD shall be greater than either of the interior opposite angles CBA, BAC.

Construction: Bisect AC in E (Proposition 10), join BE, and produce BE to F, making EF equal to EB (Proposition 3), and join FC.

Proof: Because AE is equal to EC, and BE to EF, the two sides AE, EB are equal to the two sides CE, EF, each to each; and the angle AEB is equal to the angle CEF, because they are opposite vertical angles (Proposition 15); therefore the base AB is equal to the base CF, and the tri-

angle AEB to the triangle CEF, and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite (Proposition 4): therefore the angle BAE is equal to the angle ECF. But the angle ECD is greater than the angle ECF. Therefore the angle ECD is greater



than the angle BAE. That is to say, the angle ACD is greater than the angle BAC.

In the same manner if BC be bisected and the line AC be produced to G, it may be proved that the angle BCG — that is, the angle ACD — is greater than the angle ABC (Proposition 15).

PROPOSITION 17. THEOREM

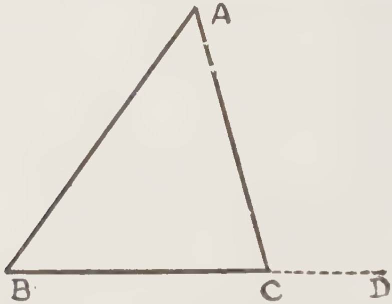
Any two angles of a triangle are together less than two right angles.

Let ABC be any triangle; any two of its angles are together less than two right angles.

Construction: Produce BC to D.

Proof: Then because ACD is the exterior angle of the triangle ABC it is greater than the interior and opposite angle ABC (Proposition 16). To each of these add the angle ACB. Therefore the angles ACD, ACB are greater

than the angles ABC, ACB. But the angles ACD, ACB are together equal to two right angles (Proposition 13). Therefore the angles ABC, ACB are together less than two right angles.

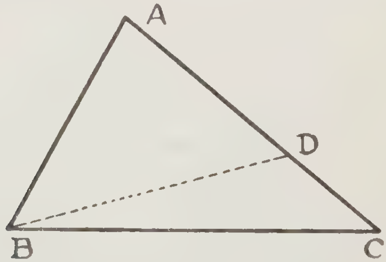


In like manner it may be proved that the angles BAC, ACB, as also the angles CAB, ABC, are together less than two right angles.

Proposition 18. Theorem

The greater side of every triangle has the greater angle opposite to it.

Let ABC be a triangle, of which the side AC is greater than the side AB; then the angle ABC is also greater than the angle ACB.



Construction: Since AC is greater than AB, make AD equal to AB (Proposition 3), and join BD.

Proof. Then, because ADB is the exterior angle of the triangle BDC, it is greater than the interior and opposite angle DCB (Proposition 16). But the angle ADB is equal to the angle ABD (Proposition 5), because the side AD is equal to the side AB. Therefore the angle ABD is also greater than the angle ACB. Much more, then, is the angle ABC greater than the angle ACB.

QUESTIONS AND EXERCISES

1. Write out in full the proof of the second part of Proposition 16.

2. If in Proposition 16 the points A and F be joined, prove that AF equals BC.

3. Prove that the angles at the base of an isosceles triangle are together less than two right angles.

4. Prove that a triangle can have but one right angle.

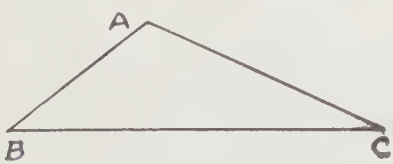
5. Compare Proposition 18 with Proposition 5.

Lesson No. 11

Proposition 19. Theorem

The greater angle of every triangle is subtended by the greater side or has the greater side opposite to it.

Let ABC be a triangle, of which the angle ABC is greater than the angle ACB; then the side AC is also greater than the side AB.



Proof: For if not, AC is either equal to AB or less than AB.

But AC is not equal to AB, for then the angle ABC would be equal to the angle ACB (Proposition 5); but it is

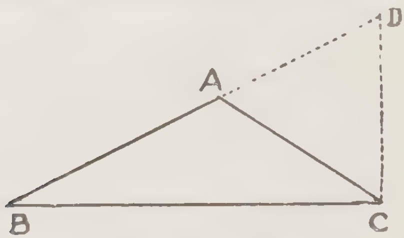
not; therefore AC is not equal to AB. Neither is AC less than AB, for then the angle ABC would be less than the angle ACB (Proposition 18); but it is not; therefore AC is not less than AB. And it has been proved that AC is not equal to AB.

Therefore AC is greater than AB.

Proposition 20. Theorem

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle; any two sides of it are together greater than the third side; namely, BA, AC greater than BC; AB, BC greater than AC; and BC, CA greater than AB.



Construction: Produce BA to the point D, making AD equal to AC (Proposition 3), and join DC.

Proof: Then, because AD is equal to AC, the angle ADC is equal to the angle ACD (Proposition 5). But the angle BCD is greater than the angle ACD. Therefore the angle BCD is greater than the angle BDC. And because the angle BCD of the triangle BCD is greater than its angle BDC, and that the greater angle is subtended by the greater side (Proposition 19), therefore the side BD is greater than the side BC. But BD is equal to BA and AD; that is, to BA and AC.

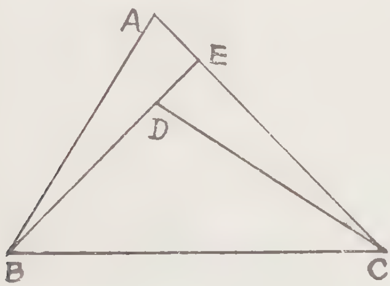
Therefore BA and AC are greater than BC.

In the same manner it may be proved that AB and BC are greater than AC, and BC and CA greater than AB.

Proposition 21. Theorem

If from the ends of a side of a triangle there be drawn two straight lines to a point within the triangle, these shall together be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a triangle, and from the points B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to the point D within the triangle: BD, DC shall together be less than the other two sides BA and AC of the triangle, but shall contain an angle BDC greater than the angle BAC.



Construction: Produce BD to E.

Proof: Because two sides of a triangle are greater than the third side, the two sides BA and AE of the triangle ABE are greater than the side BE (Proposition 20). To each of these add EC. Therefore BA and AC are greater than BE and EC. Again, the two sides CE and ED of the triangle CED are greater than the third side CD (Proposition 20). To each of these add DB. Therefore CE and EB are greater than CD and DB. But it has been proved that BA and AC are greater than BE and EC. Much more then are BA and AC greater than BD and DC.

Again, because the exterior angle of any triangle is greater than the interior opposite angle, the exterior angle BDC of the triangle CDE is greater than the angle CED (Propo-

sition 16). For the same reason, the exterior angle CEB of the triangle ABE is greater than the angle BAE. Much more then is the angle BDC greater than the angle BAC.

QUESTIONS AND EXERCISES

1. Prove that the hypotenuse is the greatest side of a right-angled triangle.

2. Prove that the diagonal of a square is greater than its side.

3. In Proposition 20 write out in full the proof that AC and CB are greater than AB.

4. Prove that any three sides of a quadrilateral are greater than the fourth side.

5. Construct a triangle, having given the base, one of the angles at the base, and the sum of the two sides.

Note. — The construction for this exercise depends on Proposition 23.

Lesson No. 12

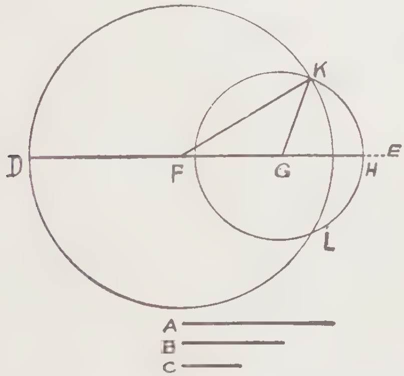
Proposition 22. Problem

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

Let A, B, C be the three given straight lines, of which any two whatever are greater than the third namely, A and B greater than C; A and C greater than B; and B and C greater than A. It is required to make a triangle of which the sides shall be equal to A, B, C, each to each.

Construction: Take a straight line DE terminated at the point D, but unlimited toward E, and make DF equal to A, FG equal to B, and GH equal to C (Proposition 3). From the center F, at the distance FD, describe the circle DKL. From the center G, at the distance GH, describe the circle HKL. Join KF, KG. The triangle KFG shall have its sides equal to the three straight lines A, B, C.

Proof: Because the point F is the center of the circle DKL, FD is equal to FK. But FD is equal to A. Therefore FK is equal to A. Again, because the point G is the center of the circle HLK, GH is equal to GK. But GH is



equal to C. Therefore GK is equal to C. And FG is equal to B. Therefore the three straight lines KF, FG, GK are equal to the three straight lines A, B, C.

Therefore the triangle KFG has its three sides KF, FG, GK equal to the three given straight lines A, B, C.

Note. — If any two of the three lines A, B, C were not greater than the third, the construction would fail. Thus, if B and C were not greater than A, then FH would not be greater than FD, and the circle described from center F, with radius FD, would therefore pass through the point H, or fall beyond it. In either case one circle would lie wholly within the other and there would be no point of intersection. In like manner the construction would fail if A and C were not greater than B, for then each circle would lie wholly without the other.

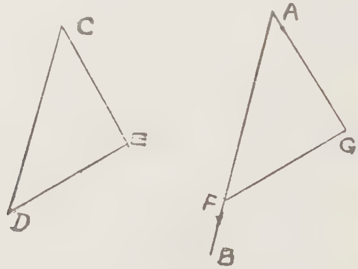
Proposition 23. Problem

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line and A the given point in it, and DCE the rectilineal angle; it is required to make

at the given point A, in the given straight line AB, an angle equal to the given rectilineal angle DCE.

Construction: In CD, CE take any points D, E, and join DE. Make the triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, EC; so that AF shall be equal to CD, AG to CE, and FG to DE (Proposition 22). The angle FAG shall be equal to the angle DCE.



Proof: Because FA, AG are equal to DC, CE, each to each, and the base FG equal to the base DE, therefore the angle FAG is equal to the angle DCE (Proposition 8).

Therefore at the given point A in the given straight line AB, the angle FAG has been made equal to the given rectilineal angle DCE.

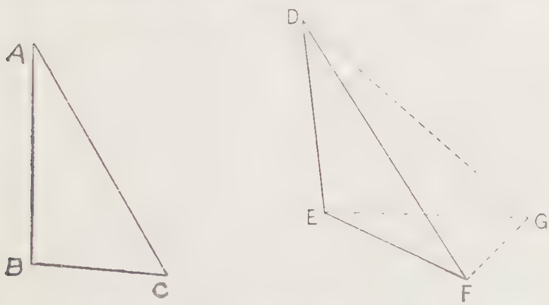
Proposition 24. Theorem

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each; namely, AB to DE and AC to DF, but the angle BAC greater than the angle EDF; then the base BC shall be greater than the base EF.

Construction: Of the two sides DE, DF, let DE be the side which is not greater than the other. At the point D in the straight line DE make the angle EDG equal to the angle BAC (Proposition 23), and make DG equal to AC or DF (Proposition 3) and join EG, GF.

Proof: Because AB is equal to DE and AC to DG, the two sides BA, AC are equal to the two sides ED, DG, each to each; and the angle BAC is equal to the angle EDG; therefore the base BC is equal to the base EG (Proposition 4). And because DG is equal to DF, the angle DGF is



equal to the angle DFG (Proposition 5). But the angle DGF is greater than the angle EGF. Therefore the angle DFG is greater than the angle EGF. Much more then is the angle EFG greater than the angle EGF. And because the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater angle is subtended by the greater side (Proposition 19), therefore the side EG is greater than the side EF. But EG was proved equal to BC. Therefore BC is greater than EF.

QUESTIONS AND EXERCISES

1. What is the reason in Proposition 22 for stating that the sum of every two of the given lines must be greater than the third?

2. Under what conditions would the circles not intersect?

3. Show how to make an angle double of a given angle.

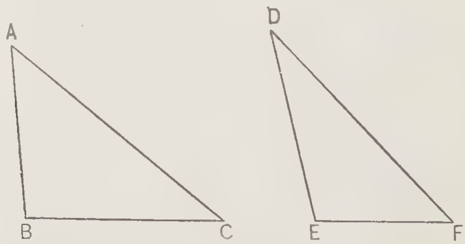
4. Construct a triangle having given two sides and the angle between them.

Lesson No. 13

Proposition 25. Theorem

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, then the angle contained by the sides of that triangle which has the greater base shall be greater than the angle contained by the sides equal to them, of the other triangle.

Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each, namely, AB to DE, and AC to DF, but the base BC greater than the base EF; then the angle BAC shall be greater than the angle EDF.



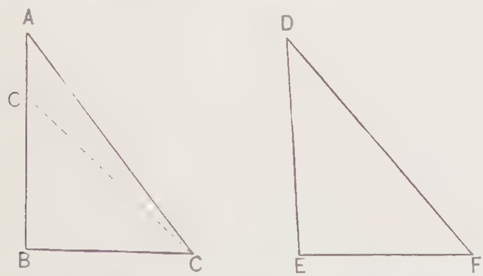
Proof: For if not, the angle BAC is either equal to the angle EDF or less than the angle EDF. But the angle BAC is not equal to the angle EDF, for then the base BC would be equal to the base EF (Proposition 4); but it is not; therefore the angle BAC is not equal to the angle EDF. Neither is the angle BAC less than the angle EDF, for then the base BC would be less than the base EF (Proposition 24); but it is not; therefore the angle BAC is not less than the angle EDF. And it has been proved that the angle BAC is not equal to the angle EDF.

Therefore the angle BAC is greater than the angle EDF

PROPOSITION 26. THEOREM

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the side which is adjacent to the angles that are equal, or a side which is opposite to one of the equal angles; then shall the other sides be equal, each to each; and also the third angle of the one equal to the third angle of the other.

Let ABC, DEF be two triangles, which have the angles ABC, BCA equal to the angles DEF, EFD, each to each, namely ABC to DEF, and BCA to EFD; and let them have also one side equal to one side.



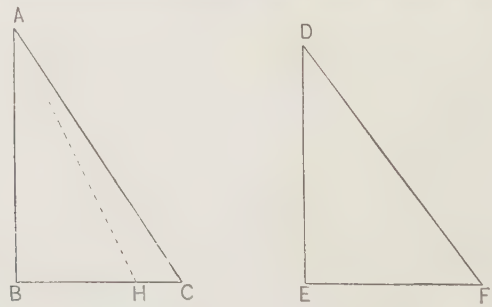
And first let those sides be equal which are adjacent to the equal angles in the two triangles, namely BC to EF. Then the other sides shall be equal, each to each, namely, AB to DE, and AC to DF, and the third angle BAC equal to the third angle EDF.

Construction: For if AB be not equal to DE, one of them must be greater than the other. Let AB be the greater, and make BG equal to DE (Proposition 3) and join GC.

Proof: Then because GB is equal to DE, and BC to EF, the two sides GB and BC are equal to the two sides DE and EF, each to each; and the angle GBC is equal to the angle DEF; therefore the base GC is equal to the base DF, and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are

opposite (Proposition 4); therefore, the angle GCB is equal to the angle DFE. But the angle DFE is equal to the angle ACB. Therefore the angle GCB is equal to the angle ACB, the less to the greater, which is impossible. Therefore AB is not unequal to DE; that is, it is equal to it. And since AB is equal to DE and BC equal to EF, therefore the two sides AB and BC are equal to the two sides DE and EF, each to each, and the angle ABC is equal to the angle DEF; therefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF (Proposition 4).

Next, let a side which is opposite to one of the equal angles in one triangle be equal to a side similarly situated in the other; namely, the side AB which is opposite the angle ACB equal to the side DE which is opposite the angle DFE (equal to ACB). Then likewise in this case the other sides shall be equal, each to each, namely, BC to EF, and AC to DF; and also the third angle BAC equal to the third angle EDF.



Construction: For if BC be not equal to EF, one of them must be greater than the other. Let BC be the greater, and make BH equal to EF (Proposition 3) and join AH.

Proof: Then because BH is equal to EF, and AB to DE, the two sides AB, BH are equal to the two sides DE, EF, each to each; and the angle ABH is equal to the angle DEF; therefore the base AH is equal to the base DF, and the triangle ABH to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides

are opposite (Proposition 4); therefore the angle BHA is equal to the angle EFD. But the angle EFD is equal to the angle BCA. Therefore the angle BHA is equal to the angle BCA (Axiom 1); that is, the exterior angle BHA of the triangle AHC is equal to the interior and opposite angle BCA, which is impossible (Proposition 16). Therefore BC is not unequal to EF; that is, it is equal to it. And since BC is equal to EF and AB equal to DE, therefore the two sides AB and BC are equal to the two sides DE and EF, each to each, and the angle ABC is equal to the angle DEF; therefore, the base AC is equal to the base DF, and the third angle BAC to the third angle EDF (Proposition 4).

NOTE. — It is evident also that the triangles are equal in all respects, as in Propositions 4 and 8.

QUESTIONS AND EXERCISES

1. Name four sets of conditions under which pairs of triangles may be proven equal to one another in all respects. What propositions deal with these four sets of conditions respectively?

2. Name the conditions under which (1) the sides, (2) the angles, of triangles are equal to one another. What propositions deal with these conditions respectively?

3. Name the conditions under which (1) the sides, (2) the angles, of triangles are unequal to one another. What propositions deal with these conditions respectively.

Lesson No. 14

DEFINITIONS

10. If two straight lines in the same plane be such that when produced indefinitely they do not meet, they are said to be parallel.

11. A parallelogram is a quadrilateral both pairs of whose opposite sides are parallel.

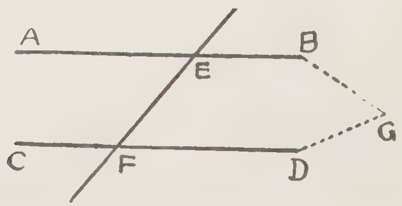
12. The straight line joining either pair of opposite angles of a quadrilateral is called a diagonal

PROPOSITION 27. THEOREM

If a straight line falling on two other straight lines make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

Let the straight line EF, which falls on the two straight lines AB, CD, make the alternate angles AEF, EFD equal to one another; then AB shall be parallel to CD.

Construction: For if it be not parallel, AB and CD being produced will meet either toward B and D, or toward A and C. Let them be produced and meet toward B and D in the point G.



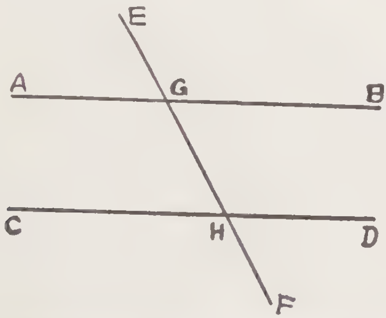
Proof: Then because GEF is a triangle its exterior angle AEF is greater than the interior opposite angle EFG (Proposition 16). But the angle AEF is equal to the angle EFG. Therefore AEF is both greater than and equal to EFG, which is impossible. Therefore AB and CD being produced, do not meet toward B and D. In like manner, it may be proved that they do not meet toward A and C. But those straight lines which, being produced ever so far both ways, do not meet, are parallel (Definition 9). Therefore AB is parallel to CD.

PROPOSITION 28. THEOREM

If a straight line falling on two other straight lines make the exterior angle equal to the interior and opposite angle on the same side of the line, or make the interior angles on the same side together equal to two right angles, the two straight lines shall be parallel to one another.

I. Let the straight line EF, which falls on the two straight lines AB, CD, make the exterior angle EGB equal to the interior and opposite angle GHD on the same side of EF; then AB shall be parallel to CD.

Proof: Because the angle EGB is equal to the angle AGH (Proposition 15), therefore the angle AGH is equal to the angle GHD. But AGH and GHD are alternate angles; therefore AB is parallel to CD (Proposition 27).



II. Let the straight line EF fall on the two straight lines AB, CD, and make the interior angles BGH, GHD on the same side of EF together equal to two right angles; then AB shall be parallel to CD.

Proof: Because the angles BGH, GHD are together equal to two right angles, and because the angles AGH, BGH are also together equal to two right angles (Proposition 13), therefore the angles AGH and BGH are together equal to the angles BGH and GHD. Take away the common angle BGH; then the remaining angle AGH is equal to the remaining angle GHD; and they are alternate angles; therefore AB is parallel to CD (Proposition 27).

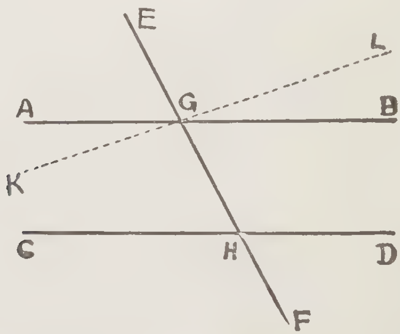
AXIOM

6. Two straight lines which cut one another cannot both be parallel to a third straight line.

PROPOSITION 29. THEOREM

If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

Let the straight line EF fall on the two parallel straight lines AB, CD; then the alternate angles AGH, GHD shall be equal to one another, and the exterior angle EGB shall be equal to the interior and opposite angle on the same side, GHD, and the two interior angles on the same side, BGH, GHD, shall be together equal to two right angles.



Construction: For if the angle AGH be not equal to the angle GHD, one of them must be greater than the other; let the angle AGH be the greater. Then, since AGH is greater than GHD, at the point G in the straight line GH make the angle KGH equal to the angle GHD (Proposition 23). Produce KG to L.

Proof: Then since GH falls upon the two straight lines KL and CD and makes the angle KGH equal to the alternate angle GHD; therefore KL is parallel to CD. But AB is parallel to CD. Therefore KL and AB, which cut one another, are both parallel to CD. But this is impossible

(Axiom 6). Therefore the angle *AGH* is not unequal to the angle *GHD*; that is, it is equal to it; that is, the alternate angles are equal.

But the angle *AGH* is equal to the angle *EGB* (Proposition 15). Therefore the angle *EGB* is equal to the angle *GHD*; that is, the exterior angle equal to the interior and opposite on the same side.

And since *EGB* is equal to *GHD*, to each of these add the angle *BGH*. Therefore the angles *EGB* and *BGH* are equal to the angles *BGH*, *GHD*. But the angles *EGB*, *BGH* are together equal to two right angles (Proposition 13). Therefore the angles *BGH*, *GHD* are together equal to two right angles; that is, the two interior angles on the same side of the line are equal to two right angles.

QUESTIONS AND EXERCISES

1. When are two straight lines parallel?

2. Of what proposition is 29 the converse?

3. Prove that the diagonal of a parallelogram makes equal angles with opposite sides.

4. Prove that all the angles of a parallelogram are together equal to four right angles.

5. Prove that the perpendicular lines drawn to parallel lines are themselves parallel.

Lesson No. 15

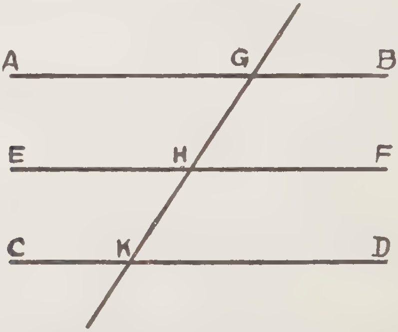
Proposition 30. Theorem

Straight lines which are parallel to the same straight line are parallel to each other.

Let *AB* and *CD* be each of them parallel to *EF*; then *AB* shall be parallel to *CD*.

Construction: Let the straight line *GHK* cut *AB*, *EF*, and *CD*.

Proof: Because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal to the angle GHF (Proposition 29). Again, because GK cuts the parallel straight lines EF, CD, the angle GHF is equal to the angle GKD (Propo-

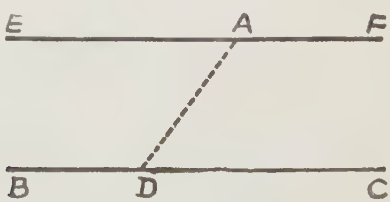


sition 29). And it was proved that the angle AGK is equal to the angle GHF. Therefore the angle AGK is equal to the angle GKD; and they are alternate angles; therefore AB is parallel to CD (Proposition 27).

Proposition 31. Problem

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point and BC the given straight line; it is required to draw a straight line through the point A parallel to the straight line BC.



Construction: In BC take any point D and join AD; at the point A in the straight line AD, make the angle DAE

equal to the angle ADC (Proposition 23), and produce the straight line EA to F. EF shall be parallel to BC.

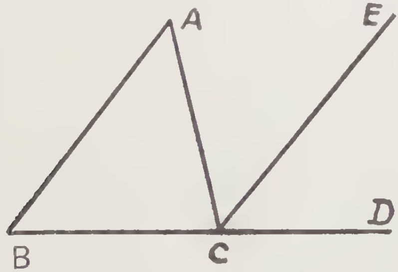
Proof: Because the straight line AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC equal to one another, EF is parallel to BC (Proposition 27).

Therefore the straight line EAF is drawn through the given point A, parallel to the given straight line BC.

Proposition 32. Theorem

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D; then the exterior angle ACD shall be equal to the two interior and opposite angles CAB, ABC; and the three interior angles of the triangle, namely, ABC, BCA, CAB, shall be equal to two right angles.



Construction: Through the point C draw CE parallel to AB (Proposition 31).

Proof: Because AB is parallel to CE, and AC falls on them, the alternate angles BAC, ACE are equal (Proposition 29). Again, because AB is parallel to CE, and BD falls on them, the exterior ECD is equal to the interior and opposite angle ABC (Proposition 29). But the angle ACE was

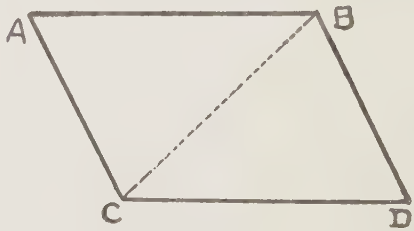
shown to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB, ABC.

Since ACD is equal to CAB and ABC together, to each of these equals add the angle ACB; therefore the angles ACD, ACB are equal to the three angles CAB, ABC, and ACB. But the angles ACD, ACB are equal to two right angles (Proposition 13); therefore, also, the angles CAB, ABC, and ACB are equal to two right angles.

Proposition 33. Theorem

The straight lines which join the extremities of two equal and parallel straight lines toward the same parts are also themselves equal and parallel.

Let AB and CD be equal and parallel straight lines, and let them be joined toward the same parts by the straight lines AC and BD; then AC and BD shall be equal and parallel.



Construction: Join BC.

Proof: Because AB is parallel to CD and BC meets them, the alternate angles ABC, BCD are equal (Proposition 29). And because AB is equal to CD and BC is common to the two triangles ABC, DCB, the two sides AB and BC are equal to the two sides DC and CB, each to each; and the angle ABC has been proved to be equal to the angle BCD; therefore the base AC is equal to the base BD, and the triangle ABC to the triangle BCD, and the other angles

to the other angles, each to each, to which the equal sides are opposite (Proposition 4); therefore the angle ACB is equal to the angle CBD.

And because the straight line BC meets the two straight lines AC and BD and makes the alternate angles ACB, CBD equal to one another, AC is parallel to BD (Proposition 27). And AC was shown to be equal to BD.

Therefore AC and BD are both equal and parallel.

QUESTIONS AND EXERCISES

1. Show how to draw through a given point a straight line, making with a given straight line an angle equal to a given angle.

2. Prove that each of the angles of an equilateral triangle is equal to two thirds of a right angle.

3. Show how to trisect a right angle.

4. Can you do Proposition 31 without using Proposition 23?

5. In a right-angled isosceles triangle what is the size of each of the base angles?

6. Prove that the angles of any quadrilateral figure are together equal to four right angles.

7. If all the interior angles of a rectilineal figure are equal to sixteen right angles, how many sides has it?

8. Divide a right-angled triangle into two isosceles triangles.

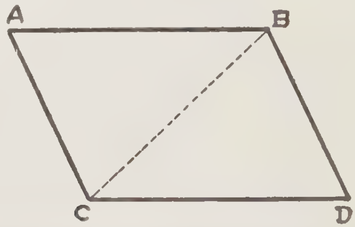
Lesson No. 16

Proposition 34. Theorem

The opposite sides and angles of a parallelogram are equal to one another and the diagonal bisects it — that is, divides it into two equal parts.

Let $ACBD$ be a parallelogram of which BC is a diagonal; the opposite sides and angles of the parallelogram shall be equal to one another, and the diagonal BC shall bisect it.

Proof. Because AB is parallel to CD and BC meets them, the alternate angles ABC, BCD are equal to one another (Proposition 29). And because AC is parallel to BD and BC meets them, the alternate angles ACB, CBD are equal to one another (Proposition 29). Therefore the two triangles ABC, BCD have two angles ABC, BCA in the one equal to two angles DCB, CBD in the other, each to each, and one side BC, which is adjacent to their equal angles, is common to the two triangles; therefore their other sides are equal, each to each, and the third angle of the one is equal to the third angle of the other; that is to say, the side AB is equal to the side CD, and the side AC is equal to the side BD, and the angle BAC is equal to the angle CDB (Proposition 26).



And because the angle ABC is equal to the angle BCD and the angle CBD to the angle ACB, therefore the whole angle ABD is equal to the whole angle ACD. And the angle BAC has been shown to be equal to the angle CDB.

Therefore the opposite sides and angles of the parallelogram are equal to one another.

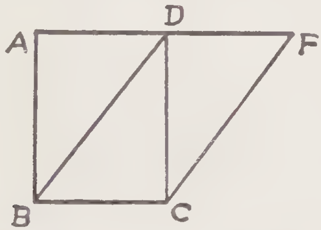
Also the diagonal bisects the parallelogram. For AB being equal to CD and BC common, the two sides AB, BC are equal to the two sides DC, CB, each to each; and the angle ABC has been proved equal to the angle BCD; therefore the triangle ABC is equal to the triangle BCD (Proposition 4).

Therefore the diagonal BC divides the parallelogram $ACDB$ into two equal parts.

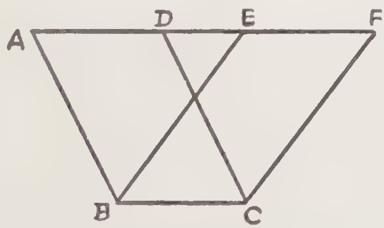
Proposition 35. Theorem

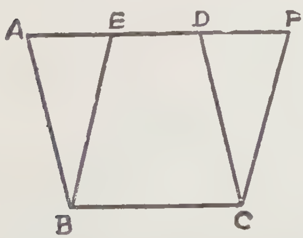
Parallelograms on the same base and between the same parallels are equal to one another.

Let the parallelograms *ABCD*, *EBCF* be on the same base *BC* and between the same parallels *AF*, *BC*; then the parallelogram *ABCD* shall be equal to the parallelogram *EBCF*.



Proof: If the sides *AD*, *EF* of the parallelograms *ABCD*, *EBCF*, opposite to the base *BC*, be terminated in the same





point *D*, it is plain that each of the parallelograms is double the triangle *BDC* (Proposition 34), and that they are therefore equal to one another.

But if the sides AD. EF, opposite to the base BC of the parallelograms $ABCD$, $EBCF$, be not terminated in the same point, then, because $ABCD$ is a parallelogram, AD is equal to BC (Proposition 34). For a similar reason EF is equal to BC; therefore AD is equal to EF; therefore the whole, or the remainder, AE, is equal to the whole, or the remainder, DF. And AB is equal to DC (Proposition 34); therefore the two EA, AB are equal to the two FD, DC, each to each; and the interior angle EAB is equal to the exterior angle FDC (Proposition 29); therefore the base EB is equal to the base FC, and the triangle EAB is equal to the triangle FDC (Proposition 4). Take the triangle FDC from the trapezium $ABCF$, and from the same trapezium take the triangle EAB, and the remainders are equal; that is, the parallelogram $ABCD$ is equal to the parallelogram $EBCF$.

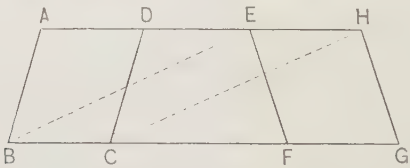
Proposition 36. Theorem

Parallelograms on equal bases, and between the same parallels, are equal to one another.

Let $ABCD$, $EFGH$ be parallelograms on equal bases BC, FG, and between the same parallels AH, BG: then the parallelogram $ABCD$ shall be equal to the parallelogram $EFGH$.

Construction: Join BE, CH.

Proof: Then, because BC is equal to FG, and FG to EH (Proposition 34), therefore BC is equal to EH; and they



are parallels (Hyp.), and joined toward the same parts by the straight lines BE, CH. But straight lines which join the extremities of equal and parallel straight lines toward

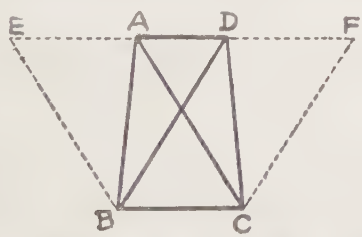
the same parts are themselves equal and parallel (Proposition 33). Therefore *BE*, *CH* are both equal and parallel. Therefore *EBCH* is a parallelogram (Definition 11). And it is equal to *ABCD*, because they are on the same base *BC*, and between the same parallels *BC* and *AH* (Proposition 35).

For a similar reason the parallelogram *EFGH* is equal to the same *EBCH*. Therefore the parallelogram *ABCD* is equal to the parallelogram *EFGH*.

Proposition 37. Theorem

Triangles on the same base, and between the same parallels, are equal.

Let the triangles *ABC*, *DBC* be on the same base *BC*, and between the same parallels *AD*, *BC*; then the triangle *ABC* shall be equal to the triangle *DBC*.



Construction: Produce *AD* both ways to the points *EF*; through *B* draw *BE* parallel to *CA*, and through *C* draw *CF* parallel to *BD* (Proposition 31).

Proof: Then each of the figures *EBCA*, *DBCF* is a parallelogram; and *EBCA* is equal to *DBCF*, because they are on the same base *BC*, and between the same parallels *BC*, *EF* (Proposition 35). And the triangle *ABC* is half of the parallelogram *EBCA*, because the diagonal *AB* bisects it (Proposition 34), and the triangle *DCB* is half of the parallelogram *DBCF*, because the diagonal *DC* bisects it (Proposition 34). But the halves of equal things are equal. Therefore the triangle *ABC* is equal to the triangle *DBC*.

QUESTIONS AND EXERCISES

1. Prove that if one angle of a parallelogram be a right angle, all its angles are right angles.

2. Prove that if the opposite angles of a quadrilateral figure be equal to one another it is a parallelogram.

3. Prove that the diagonals of a parallelogram bisect each other.

4. Make a rectangle which shall be equal in area to a given parallelogram.

5. Make a rhombus which shall be equal in area to a given parallelogram.

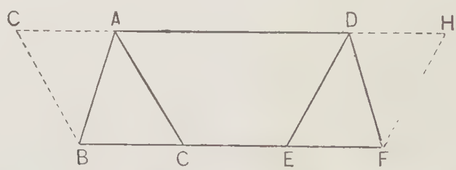
6. If a square and any other parallelogram of equal area stand on the same base, the perimeter of the square shall be less than that of the parallelogram.

Lesson No. 17

Proposition 38. Theorem

Triangles on equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be on equal bases BC, EF, and between the same parallels BF, AD: then the triangle ABC shall be equal to the triangle DEF.



Construction: Produce AD both ways to the points G, H; through B draw BG parallel to CA, and through F draw FH parallel to ED (Proposition 31).

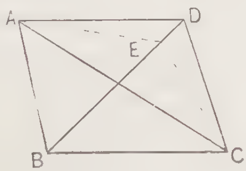
Proof: Each of the figures $GBCA$ and $DEFH$ is a parallelogram (definition). And they are equal to one another,

because they are on equal bases BC, EF, and between the same parallels, BF and GH (Proposition 36). And the triangle ABC is half of the parallelogram $GBCA$, because the diagonal AB bisects it (Proposition 34); and the triangle DEF is half of the parallelogram $DEFH$, because the diagonal DF bisects it. But the halves of equal things are equal. Therefore the triangle ABC is equal to the triangle DEF.

Proposition 39. Theorem

Equal triangles on the same base, and on the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be on the same base BC, and on the same side of it: then they shall be between the same parallels.



Construction: Join AD.

Proof: AD shall be parallel to BC. For if it is not, through A draw AE parallel to BC (Proposition 31) and join EC.

The triangle ABC is equal to the triangle EBC, because they are on the same base BC, and between the same parallels BC, AE (Proposition 37). But the triangle ABC is given equal to the triangle DBC. Therefore also the triangle DBC is equal to the triangle EBC, the greater to the less, which is impossible. Therefore AE is not parallel to BC.

In the same manner it can be demonstrated, that no other line through A but AD is parallel to BC. Therefore AD is parallel to BC.

QUESTIONS AND EXERCISES

1. Prove Proposition 38 when the vertices of the triangles fall upon the same point.

2. Prove that the four triangles into which a parallelogram is divided by its diagonals are equal to one another.

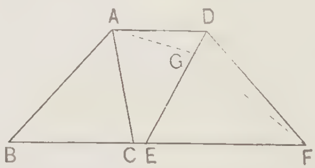
3. Prove that the straight line which joins the middle points of two sides of a triangle is parallel to the base.

Lesson No. 18

Proposition 40. Theorem

Equal triangles, on equal bases in the same straight line, and toward the same parts, are between the same parallels.

Let the equal triangles ABC and DEF be on equal bases BC, EF, in the same straight line BF, and toward the same parts: then they shall be between the same parallels.



Construction: Join AD.

Proof: AD shall be parallel to BF. For if it is not, through A draw AG parallel to BF (Proposition 31) and join GF.

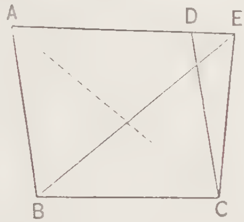
The triangle ABC is equal to the triangle GEF, because they are on equal bases BC, EF, and between the same parallels (Proposition 38). But the triangle ABC is given equal to the triangle DEF. Therefore also the triangle DEF is equal to the triangle GEF, the greater to the less, which is impossible. Therefore AG is not parallel to BF.

In the same manner it can be demonstrated that no other line through A but AD is parallel to BF. Therefore AD is parallel to BF.

Proposition 41. Theorem

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.

Let the parallelogram $ABCD$ and the triangle EBC be on the same base BC, and between the same parallels BC, AE: then the parallelogram $ABCD$ shall be double of the triangle EBC.



Construction: Join AC.

Proof: The triangle ABC is equal to the triangle EBC, because they are on the same base BC, and between the same parallels BC, AE (Proposition 37). But the parallelogram $ABCD$ is double of the triangle ABC, because the diagonal AC bisects the parallelogram (Proposition 34). Therefore the parallelogram $ABCD$ is also double of the triangle EBC.

QUESTIONS AND EXERCISES

1. Prove that if two triangles or two parallelograms be upon equal bases and have the same altitude, they shall be equal to one another.

2. Prove that the straight line joining the middle points of two sides of a triangle is parallel to the third side.

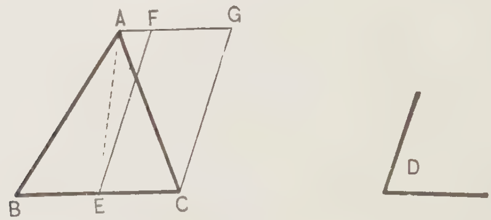
3. Prove that the line drawn parallel to any side of a triangle through the middle point of another side bisects the third side.

Lesson No. 19

Proposition 42. Problem

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle: it is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.



Construction: Bisect BC in E (Proposition 10). Join AE, and at the point E in the straight line EC make the angle CEF equal to D (Proposition 23). Through A draw AFG parallel to EC, and through C draw CG parallel to EF (Proposition 31). Then $FECG$ is a parallelogram (Definition 11).

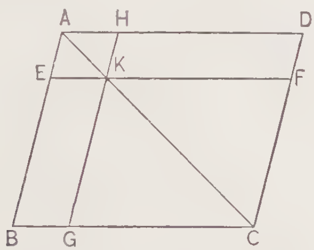
Proof: Because BE is equal to EC the triangle ABE is equal to the triangle AEC, for they are on equal bases BE, EC, and between the same parallels BC, AG (Proposition 38). Therefore the triangle ABC is double the triangle AEC. But the parallelogram $FECG$ is also double the triangle AEC, because they are on the same base EC, and between the same parallels EC, AG (Proposition 41). Therefore the parallelogram $FECG$ is equal to the triangle ABC; and it has one of its angles CEF equal to the given angle D.

Therefore a parallelogram $FECG$ has been described equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D.

Proposition 43. Theorem

The complements of the parallelograms which are about the diagonal of any parallelogram are equal to one another.

Let $ABCD$ be a parallelogram, of which the diagonal is AC; and EH, GF parallelograms about AC, that is, through which AC passes; and BK, KD the other parallelograms which make up the whole figure $ABCD$, and which are therefore called the complements. Then the complement BK shall be equal to the complement KD.



Proof: Because $ABCD$ is a parallelogram, and AC its diagonal, the triangle ABC is equal to the triangle ADC (Proposition 34). Similarly, because $AEKH$ is a parallelogram, and AK its diameter, the triangle AEK is equal to the triangle AHK. For the like reason the triangle KGC is equal to the triangle KFC. Therefore, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to the triangle KFC, the triangle AEK together with the triangle KGC is equal to the triangle AHK together with the triangle KFC. But the whole triangle ABC was proved equal to the whole triangle ADC. Therefore the remainder, the complement BK, is equal to the remainder, the complement KD.

QUESTIONS AND EXERCISES

1. Construct a square which shall be equal to a given right-angled isosceles triangle.

2. In the figure for Proposition 43 prove that all the parallelograms of the figure are equiangular.

3. Prove that if K be a point within a parallelogram $ABCD$ such that lines drawn parallel to the sides make parallelograms DK and KB equal to one another, then K is a point in the diagonal AC.

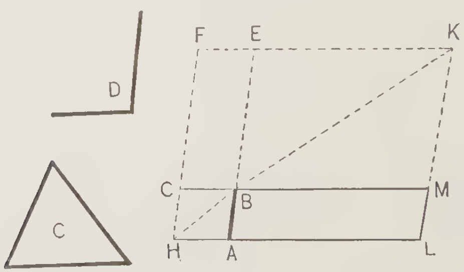
4. In the figure for Proposition 43 what other parallelograms are equal besides the pair of complements?

Lesson No. 20

Proposition 44. Problem

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle: it is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.



Construction: Make the parallelogram $BEFG$ equal to the triangle C, having the angle EBG equal to the angle D, and so that the side BE may be in the same straight line with AB (Proposition 42). Produce FG to H. Through A draw AH, parallel to BG or EF (Proposition 31), and join HB.

Then, because AH is parallel to FE, HB is not parallel to FE (Axiom 6). Therefore HB and FE will meet if pro-

duced. Let them meet in K. BL shall be the parallelogram required.

Through K draw KL, parallel to EA or FH (Proposition 31); and produce HA and GB to the points L and M.

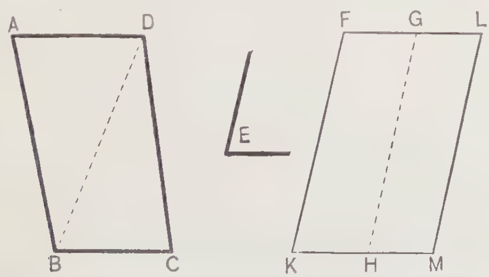
Proof: $HLKF$ is a parallelogram, of which the diagonal is HK; and AG, ME are parallelograms about HK; and LB, BF are the complements. Therefore LB is equal to BF (Proposition 43). But BF is equal to the triangle C. Therefore LB is equal to the triangle C. And because the angle GBE is equal to the angle ABM (Proposition 15), and likewise, by construction, to the angle D; therefore the angle ABM is equal to the angle D.

Therefore to the given straight line AB the parallelogram LB is applied, equal to the triangle C, and having the angle ABM equal to the angle D.

Proposition 45. Problem

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let $ABCD$ be the given rectilineal figure, and E the given rectilineal angle: it is required to describe a parallelogram equal to $ABCD$, and having an angle equal to E.



Construction: Join DB, and describe the parallelogram FH equal to the triangle ADB, and having the angle FKH equal to the angle E (Proposition 42); and to the straight line GH apply the parallelogram GM equal to the triangle

DBC, and having the angle *GHM* equal to the angle *E* (Proposition 44). The figure *FKML* shall be the parallelogram required.

Proof: Because the angle *E* is equal to each of the angles *FKH, GHM*, the angle *FKH* is equal to the angle *GHM*. Add to each of these equals the angle *KHG*; then the angles *FKH, KHG* are equal to the angles *KHG, GHM*. But *FKH, KHG* are equal to two right angles (Proposition 29); therefore, also, *KHG, GHM* are equal to two right angles. And because at the point *H* in the straight line *GH*, the two straight lines *KH, HM*, on the opposite sides of it, make the adjacent angles together equal to two right angles, *KH* is in the same straight line with *HM* (Proposition 14).

And because the straight line *HG* meets the parallels *KM, FG*, the alternate angles *MHG, HGF* are equal (Proposition 29). Add to each of these equals the angle *HGL*; then the angles *MHG* and *HGL* are equal to the angles *HGF* and *HGL*. But *MHG* and *HGL* are equal to two right angles (Proposition 29); therefore, also, *HGF* and *HGL* are equal to two right angles. Therefore *FG* is in the same straight line with *GL* (Proposition 14).

And because *KF* is parallel to *HG*, and *HG* to *ML* (by construction), therefore *KF* is parallel to *ML* (Proposition 30). And *KM, FL* are parallels, because *FG* and *KH* are the opposite sides of the parallelogram *FH*, and *FL* is in the same straight line with *FG*, and *KM* in the same straight line with *KH*. Therefore *KFLM* is a parallelogram (Definition 11).

And because the triangle *ABD* is equal to the parallelogram *HF*, and the triangle *DBC* to the parallelogram *GM*, therefore the whole rectilineal figure *ABCD* is equal to the whole parallelogram *KFLM*.

Therefore the parallelogram *KFLM* has been described equal to the given rectilineal figure *ABCD*, and having the angle *FKM* equal to the given angle *E*.

Corollary. From this it is manifest how, to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure; namely, by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle; and so on.

QUESTIONS AND EXERCISES

1. On a given straight line construct a right-angled triangle that shall be equal in area to a given triangle.

2. Show how to make a triangle equal in area to a given quadrilateral.

3. Show how to construct a rectangle equal to: (1) the difference between two triangles; (2) the difference between any two rectilineal figures.

Lesson No. 21

Proposition 46. Problem

To describe a square on a given straight line.

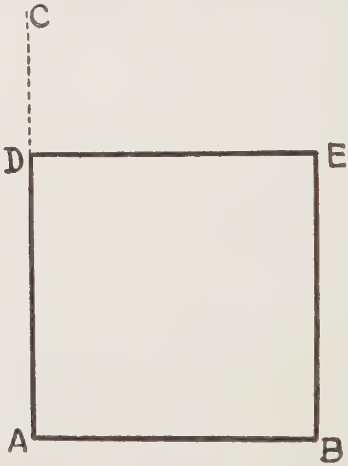
Let AB be the given straight line. It is required to describe a square on AB.

Construction: From the point A draw AC at right angles to AB (Proposition 11) and make AD equal AB (Proposition 3); through D draw DE parallel to AB, and through B draw BE parallel to AD (Proposition 31). $ADEB$ shall be a square.

Proof: For $ADEB$ is by construction a parallelogram; therefore AB is equal to DE and AD to BE (Proposition 34). But AB is equal to AD. Therefore the four straight lines BA, AD, DE, EB are equal to one another, and the parallelogram $ADEB$ is equilateral.

Likewise all its angles are right angles. For since the straight line AD meets the parallels AB, DE, the angles BAD, ADE are together equal to two right angles (Proposi-

tion 29). But BAD is a right angle; therefore, also, ADE is a right angle. But the opposite angles of parallelograms are equal (Proposition 34). Therefore each of the opposite



angles ABE, BED is a right angle. Therefore the figure $ADEB$ is rectangular, and it has been proved to be equilateral. Therefore it is a square. And it is described on the given straight line AB.

Proposition 47. Theorem

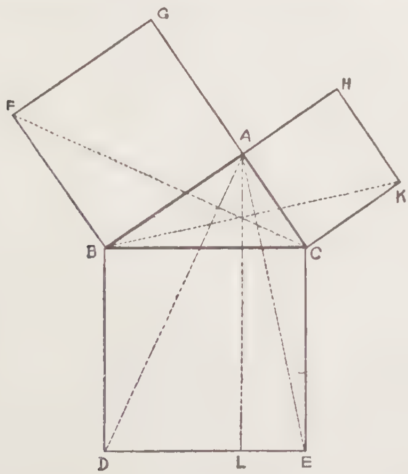
In any right-angled triangle the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC; then the square described on the side BC shall be equal to the squares described on the sides BA and AC.

Construction: On BC describe the square $BDEC$, and on BA and AC describe the squares GB and HC (Proposition 46); through A draw AL parallel to BD or CE (Proposition 31), and join AD, FC.

Proof : Because the angle BAC is a right angle, and because the angle BAG is also a right angle, then the two straight lines AC, AG, on the opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles; therefore CA is in the same straight line with AG (Proposition 14). For a similar reason AB and AH are in the same straight line.

Now the angle DBC is equal to the angle FBA, for each of them is a right angle. Add to each the angle ABC. Therefore the whole angle DBA is equal to the whole angle FBC. And because in the two triangles ABD and FBC, the two sides AB and BD are equal to the two sides FB and BC, each to each, and the angle ABD is equal to the angle FBC; therefore the base AD is equal to the base FC, and the triangle ABD to the triangle FBC (Proposition 4).



Now the parallelogram BL is double of the triangle ABD, because they are on the same base BD and between the same parallels BD and AL (Proposition 41). And the square GB is double of the triangle FBC, because they are

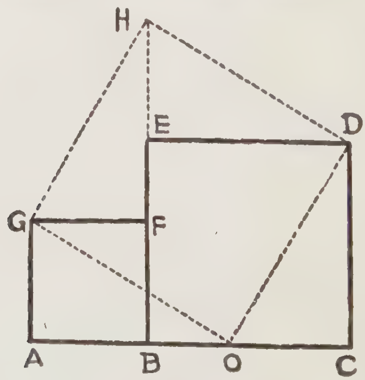
on the same base FB and between the same parallels FB, GC. But the doubles of equals are equal to one another. Therefore the parallelogram BL is equal to the square GB.

In a similar manner, by joining AE, BK, it can be demonstrated that the parallelogram CL is equal to the square HC. Therefore the whole square $BDEC$ is equal to the two squares GB, HC. And the square $BDEC$ is described on BC, and the squares GB and HC on BA and AC.

Therefore the square described on the side BC is equal to the squares described on sides BA and AC.

The following proof of this proposition is interesting:

Let BG, BD be any two squares having their bases in the same straight line. Take $CO = AB$. Join GO, OD. Make $FH = ED$ the side of the greater square. Join GH and DH. Then the four triangles AOG, ODC, DEH, GFH may all be proved equal to each other (Proposition 4).



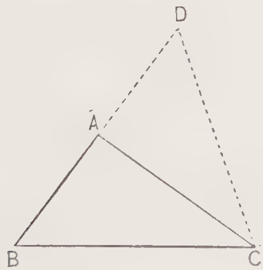
Therefore the figure $GODH$ is equal to the two squares AF and EC; and it is equilateral. Also by Proposition 32 it may be shown to be rectangular. Therefore it is a square; and it is the square described on the hypotenuse of a right-angled triangle, two of whose sides are OC and CD; that is, AB and BC.

Lesson No. 22

Proposition 48. Theorem

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described on BC, one of the sides of the triangle ABC, be equal to the squares described on the other sides, BA, AC: then the angle BAC shall be a right angle.



Construction: From the point A draw AD at right angles to AC (Proposition 11), and make AD equal to BA. Join DC.

Proof: Then, because DA is equal to BA, the square on DA is equal to the square on BA. To each of these add the square on AC. Therefore the squares on DA and AC are equal to the squares on BA and AC. But the angle DAC is a right angle; therefore the square on DC is equal to the squares on DA, AC (Proposition 47). And by our hypothesis, the square on BC is equal to the squares on BA, AC. Therefore the square on DC is equal to the square on BC. Therefore, also, the side DC is equal to the side BC.

And because the side DA is equal to the side AB (by construction), and the side AC is common to the two triangles DAC, BAC, then the two sides DA and AC are equal to the two sides BA and AC, each to each; and the base DC has been proved equal to the base BC; therefore

the angle DAC is equal to the angle BAC (Proposition 8). But by construction DAC is a right angle. Therefore, also, BAC is a right angle.

QUESTIONS AND EXERCISES

1. Prove that the squares described on equal straight lines are equal.

2. Prove that the perimeter of a square is less than that of any other equal parallelogram on the same base.

3. Prove that the diagonals of a square bisect each other at right angles.

4. The sides of a right-angled triangle are 9 and 16 inches, respectively; find the length of the hypotenuse.

5. Prove that the sum of the squares on the diagonals of a rectangle is equal to the sum of the squares on its four sides.

6. Prove that the triangle whose sides are 12, 16, and 20 inches, respectively, is a right-angled triangle.

7. Prove that the square on a given line is equal to four times the square on half the line.

8. Divide a given straight line into two parts such that the sum of their squares may be equal to a given square.

9. Divide a given square into five equal parts; namely, four right-angled triangles and a square.

10. Write out the enunciation for a proposition which is the converse of Proposition 47. Compare your enunciation with that of Proposition 48.

11. Show that if in a triangle ABC, the squares on BA and AC are greater than the square on BC, the angle BAC is acute; but if less, then that BAC is obtuse.

12. Make a square equal to the difference between two squares.